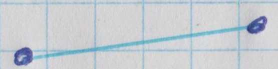
Welcome to Math 2260H - Euclidean Geometry

Euclid's <u>Elements of Geometry</u> (composed c. 300BC)
(using the free translation by Richard Fitzpatrick)

- the geometry of 2-D (plane) space [flat]

The Postulates: (Visually) [What you can do with an unmarked straightedge & a compass.]

I. 

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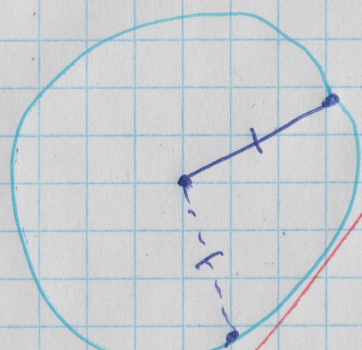
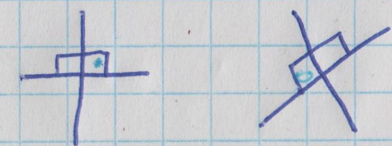
III. 

IV. 

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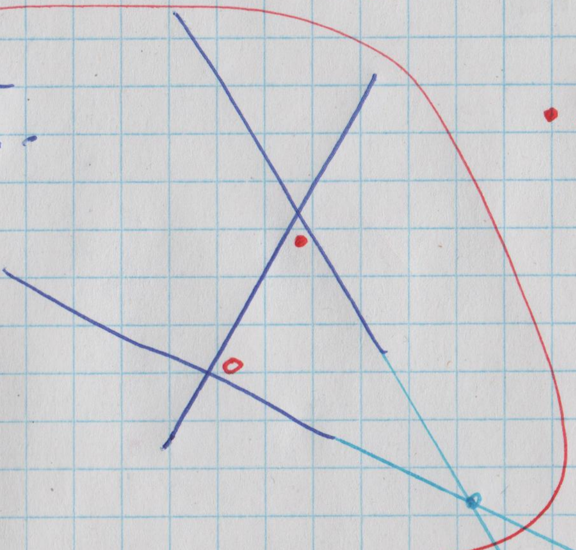
V. 

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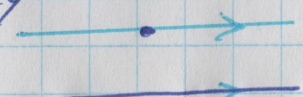
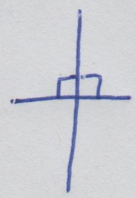
V'. (Playfair's Postulate) 

Definitions

(Taken from Book I of Richard Fitzpatrick's translation of the *Elements*.)

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with the straight-lines on itself.
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.



11. An obtuse angle is one greater than a right-angle.

12. And an acute angle (is) one less than a right-angle.
13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.
18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those

contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.
21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.
22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.
23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

Postulates

1. Let it have been postulated to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

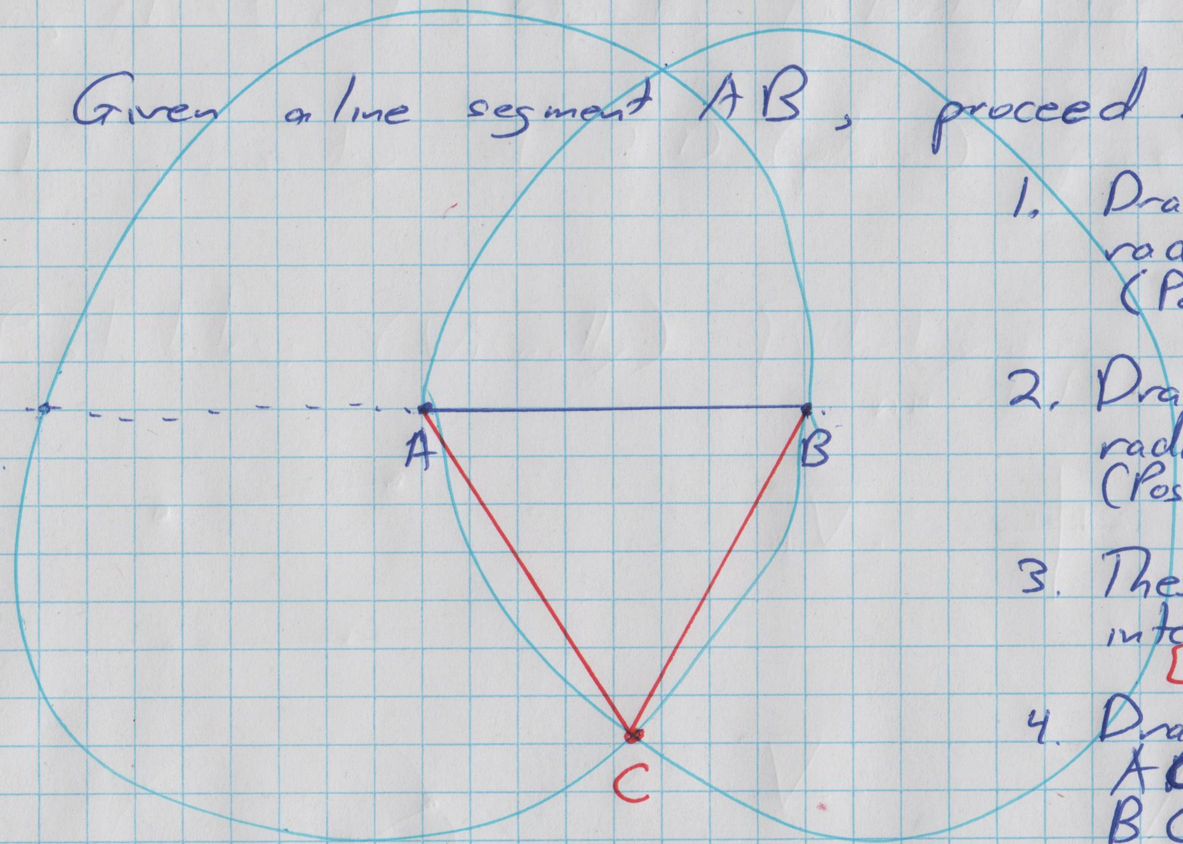
3. And if equal things are subtracted from equal things then the remainders are equal.

4. And things coinciding with one another are equal to one another.

5. And the whole [is] greater than the part.

Proposition I: One can construct, given a line segment, an equilateral triangle one of whose sides is the given line-segment.

proof: Given a line segment AB, proceed as follows:



1. Draw a circle with radius AB and centre A. (Postulate III)
2. Draw a circle with radius AB & centre B. (Postulate III)
3. These two circles must intersect in some point C. [Postulate 5.]
4. Draw the line segment AC and the line segment BC. (Post. I)

Claim: ΔABC is equilateral

1. $|AB| = |AC|$ since both are radii of the circle drawn in step 1. (By the defn of circles)
2. $|AB| = |BC|$ since both are radii of the circle drawn in step 2. (—— " ——).
3. $|AC| = |BC|$ [By Common Notion I.]

Thus ΔABC does the job. //

Q.: What's wrong with this argument?

A.: How do we know the two circles drawn in steps 1 and 2 actually intersect? We don't. (Intuitively obvious they should, but the Postulates don't justify the conclusion in step 3.)

We'll augment Euclid's Postulates with two more:

Postulate S: ("Separation") Any infinite line, any circle, and any triangle separate the plane into two regions such that joining a point in one region to a point in the other region intersects the separating line (or curve).

(This repairs the gap at step 3 in the previous proof.)

Postulate A: ("Application")

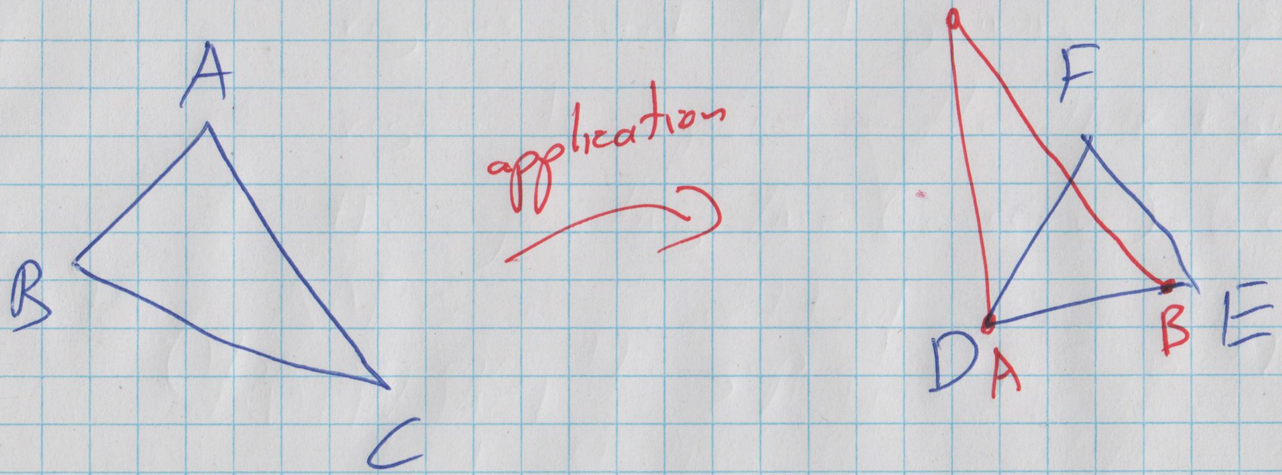
Given two triangles ΔABC & ΔDEF, it is possible to apply ΔABC to ΔDEF

(ie place a copy of ΔABC on ΔDEF)

such that vertex A falls on vertex D,

side AB falls on side DE,

and vertex C falls on the same side of side DE that vertex F does.



These repair the biggest (but not all) the holes in Euclid's Postulates.

For a really complete set of axioms for Euclidean geometry, check out the handout

<u>Hilbert's Axioms for Euclidean Geometry</u> (20+ of them)

(taken from Hilbert's book

<u>Foundations of Geometry</u>.).

Next time: We work through a bit more of Euclid's <u>Elements</u>.