Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2014

Solutions to the Quizzes

Quiz #0. Friday, 10 January, 2014 *[15 minutes]*

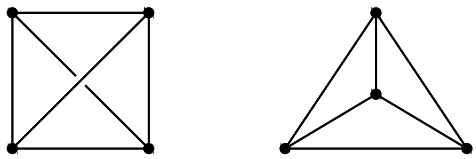
A baby plane geometry, which we'll call *Quattro*, is defined as follows:

- *Quattro* has exactly four points.
- Any two points of *Quattro* are connected by exactly one line of *Quattro*.
- Every line of *Quattro* has only two points of *Quattro* on it.

1. Draw a picture of *Quattro*. *[2]*
2. How many lines does *Quattro* have? *[1.5]*
3. How many triangles are there in *Quattro*? *[1.5]*

Bonus: What geometry do you think *Quattro* would want to be when all grown up? *[0.5]*

SOLUTIONS. 1. Here are two slightly different pictures of *Quattro*:



The one on the left is a likely first attempt for many people, and can be a little tricky to use if one forgets that the diagonal lines do not have a point of intersection in *Quattro*. The one on the right avoids this problem and makes it very easy to count the lines and triangles of *Quattro*. □

2. *Quattro* has $\binom{4}{2} = 6$ lines. □

3. *Quattro* has $\binom{4}{3} = 4$ triangles. □

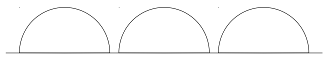
Bonus: I think *Quattro* wants to be three-dimensional as a grown-up. Note that *Quattro*'s points and lines have the same structure as the vertices and edges of a tetrahedron (three-sided pyramid) . . . ■

Quiz #1. Friday, 17 January, 2014 *[15 minutes]*

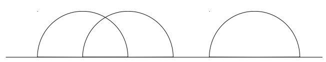
1. Three lines in the hyperbolic plane divide up the hyperbolic plane into a number of regions. What are the possible values of this number? Illustrate each possibility. *[5]*

SOLUTION. 4, 5, 6, and 7 are the possible number of regions the hyperbolic plane can be divided into by three lines, illustrated below using the Poincaré half-plane model of the hyperbolic plane. The key is to consider how the lines can intersect:

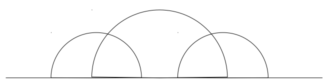
i. If the three lines do not intersect at all, they partition the hyperbolic plane into four (4) regions.



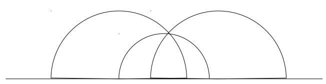
ii. If two of the three lines intersect, but neither intersects the third line, the three lines partition the hyperbolic plane into five (5) regions. [This is the case that can't happen in the Euclidean plane, where there is only one parallel line passing through a given point that is parallel to a given line.]



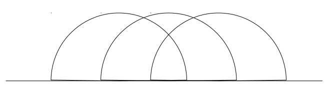
iii. If two of the lines do not intersect, but the third intersects each of the other two, the three lines partition the hyperbolic plane into six (6) regions.



iv. If all three of the lines intersect in a single point, they partition the hyperbolic plane into six (6) regions.



v. If each the three lines intersects both of the others at different points (so they form a triangle), they partition the plane into seven (7) regions.



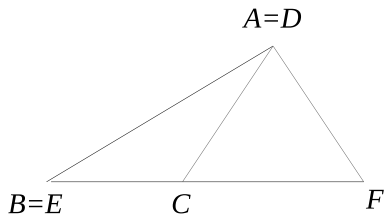
■

NOTE: You can't partition the hyperbolic [or Euclidean] plane into three or fewer regions using three different lines. A single line partitions the plane into two regions, adding a second line partitions at least one of those regions into two for a total of at least three, and adding a third line partitions at least one of those regions into two for a total of at least four. Moreover, you can't partition the hyperbolic [or Euclidean] plane into eight or more regions using three different lines. Since the best each line added could do is divide each existing region into two parts, three lines could not hope to divide the plane into more than $2^3 = 8$ regions. In the hyperbolic [and Euclidean] plane, they can't even achieve eight: in the case where two lines partition the plane into four pieces, they must intersect at a point. A third line may intersect the first two away from the point, cutting three of the four previous regions into two, for a total of seven, as in case v above. However, to cut the fourth previous region into two to make a total of eight regions, the third line would have to cross one of the previous two in a second point, which can't happen in the hyperbolic [or Euclidean] plane. (It *can* happen in spherical geometry, but some of the other cases above can't happen there.)

Quiz #3. Friday, 31 January, 2014 *[10 minutes]*

1. Show that the Angle-Side-Side (ASS) congruence criterion does not work in general. That is, find triangles $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$, $|AB| = |DE|$, and $|AC| = |DF|$, but $\triangle ABC \ncong \triangle DEF$. *[5]*

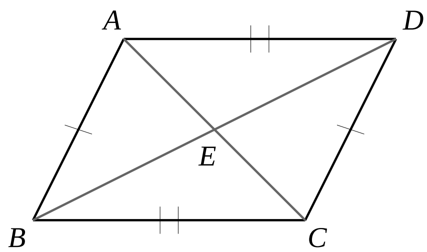
SOLUTION. The picture says it all, I think:



■

Quiz #4. Friday, 7 February, 2014 *[15 minutes]*

Suppose ABCD is a quadrilateral such that $|AB| = |CD|$ and $|AD| = |CB|$, and let E be the point of intersection of the diagonals AC and BD, as in the diagram below.



1. Show that $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$. *[2]*
2. Show that E is the midpoint of the diagonals AC and of BD. *[3]*

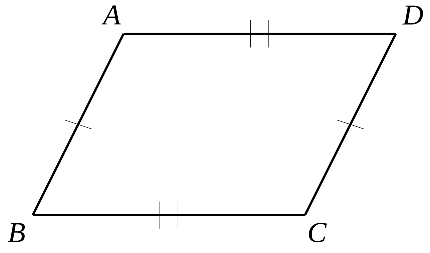
Note/Hint: You may us the Angle-Side-Angle congruence criterion for triangles in your solution to question 2.

SOLUTION TO 1. We are given that $|AB| = |CD|$ and $|AD| = |CB|$. Since $|AC| = |CA|$ – the common notions strike again! – it follows by the Side-Side-Side congruence criterion (Proposition I.8) that $\triangle ABC \cong \triangle CDA$. Similarly, since $|BD| = |DB|$, it also follows by the Side-Side-Side congruence criterion that $\triangle ABD \cong \triangle CDB$. ■

SOLUTION TO 2. Since $\triangle ABC \cong \triangle CDA$ (as shown above), it follows that $\angle BAE = \angle BAC = \angle DCA = \angle DCE$, and since $\triangle ABD \cong \triangle CDB$ (also shown above), it also follows that $\angle EBA = \angle DBA = \angle BDC = \angle EDC$. Since $|AB| = |CD|$ was given, it follows by the Angle-Side-Angle congruence criterion (Proposition I.26) that $\triangle ABE \cong \triangle CDE$. This, in turn, implies that $|AE| = |CE|$ and $|BE| = |DE|$, so E is the midpoint of both AC and BD, as required. ■

Quiz #5. Friday, 14 February, 2014 *[10 minutes]*

Suppose ABCD is a quadrilateral such that $|AB| = |CD|$ and $|AD| = |BC|$, as in the diagram below.



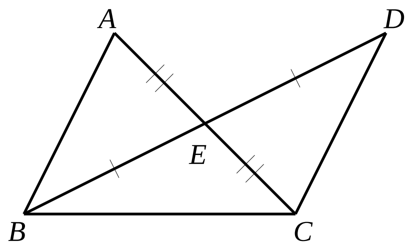
1. Show that $AB \parallel CD$ and $AD \parallel BC$. *[5]*

Solution. We are given that $|AB| = |CD|$ and $|AD| = |CB|$. Connect A to C. Since $|AC| = |CA|$, it follows by the Side-Side-Side congruence criterion (Proposition I.8) that $\triangle ABC \cong \triangle CDA$. [This was just the solution to problem 1 on Quiz #4 all over again.] It follows that $\angle BAC = \angle DCA$ and $\angle DAC = \angle BCA$.

Since $\angle BAC = \angle DCA$, Proposition I.28 [the half of the Z Theorem that does not need Postulate V] implies that $AB \parallel CD$. Similarly, since $\angle DAC = \angle BCA$, Proposition I.28 implies that $AD \parallel BC$. ■

Quiz #6. Friday, 28 February, 2014 *[10 minutes]*

Suppose A and D are points on the same side of BC and such that the point of intersection, E, of AC and BD is the midpoint of both AC and BD, as in the diagram below.



1. Show that $\triangle ABC$ and $\triangle DBC$ have equal areas. *[5]*

Solution. There are several ways to do this, among which we choose the following brutally simplemided approach.

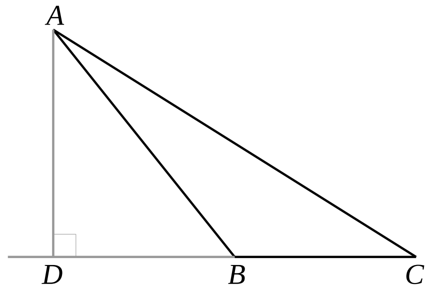
We are given that $|AE| = |CE|$ and $|BE| = |DE|$; we also have $\angle AEB = \angle CED$ since they are opposite angles. It follows by the Side-Angle-Side (SAS) congruence criterion that $\triangle AEB \cong \triangle CED$, and so these two triangles must have equal areas. Now

$$\begin{aligned} \text{area}\,(\triangle ABC) &= \text{area}\,(\triangle AEB) + \text{area}\,(\triangle BEC) \\ &= \text{area}\,(\triangle CED) + \text{area}\,(\triangle BEC) = \text{area}\,(\triangle DBC)\ , \end{aligned}$$

as desired. ■

Quiz #7. Friday, 7 March, 2014 *[10 minutes]*

Suppose $\triangle ABC$ has an obtuse angle at B and the altitude from A meets (the extension of) BC at D, as in the diagram below.



1. Show that if $|AC|^2 = |AB|^2 + 3|BC|^2$, then $|DB| = |BC|$. *[5]*

Solution. Since AD is an altitude, $\angle ADB = \angle ADC$ is a right angle, so both $\triangle ADB$ and $\triangle ADC$ are right triangles. It follows by the Pythagorean Theorem that

$$|AB|^2 = |AD|^2 + |DB|^2 \quad \text{and} \quad |AC|^2 = |AD|^2 + |DC|^2 \,,$$

and we are given that $|AC|^2 = |AB|^2 + 3|BC|^2$. Hence

$$\begin{aligned} |AB|^2 + 3|BC|^2 = |AC|^2 &= |AD|^2 + |DC|^2 = |AD|^2 + (|DB| + |BC|)^2 \\ &= |AD|^2 + |DB|^2 + 2|DB| \cdot |BC| + |BC|^2 \\ &= |AB|^2 + 2|DB| \cdot |BC| + |BC|^2 \,, \end{aligned}$$

and so

$$2|DB| \cdot |BC| = |AB|^2 + 3|BC|^2 - |AB|^2 - |BC|^2 = 2|BC|^2 \,.$$

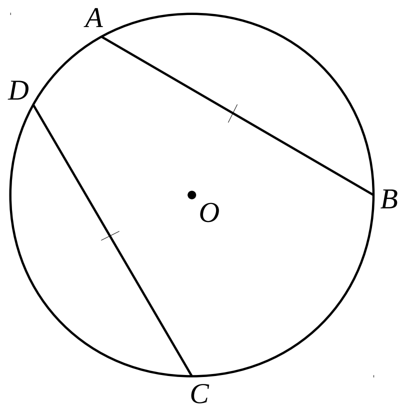
Solving for $|DB|$, we get

$$|DB| = \frac{2|BC|^2}{2|BC|} = |BC| \,,$$

as desired. ■

Quiz #8. Friday, 14 March, 2014 *[10 minutes]*

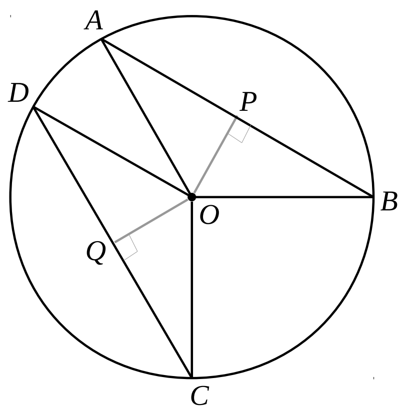
1. Suppose AB and CD are two chords of a circle with centre O such that $|AB| = |CD|$, as in the diagram below.



Show that AB and CD are the same distance from O. *[5]*

SOLUTION 1. *(Brief and correct, but not very filling.)* Connect each of A, B, C, and D to the centre O. $|AO| = |BO| = |CO| = |DO|$ since they are all radii of the circle. As it was given that $|AB| = |CD|$, it follows by the Side-Side-Side congruence criterion that $\triangle OAB \cong \triangle OCD$. Hence the corresponding sides AB and CD must be the same distance from the opposite vertex, which is O in each case. □

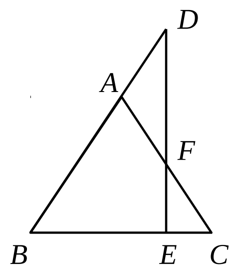
SOLUTION 2. *(More explicit, at least!)* Connect each of A, B, C, and D to the centre O. $|AO| = |BO| = |CO| = |DO|$ since they are all radii of the circle. As it was given that $|AB| = |CD|$, it follows by the Side-Side-Side congruence criterion that $\triangle OAB \cong \triangle OCD$. Let P and Q be the points where the altitudes from O meet AB and CD respectively.



OP and OQ are each parts of radii that meet the chords AB and CD at right angles, so it follows by Proposition III.3 that P and Q are the midpoints of AB and CD, respectively, and so $|AP| = \frac{1}{2}|AB| = \frac{1}{2}|CD| = |CQ|$. As noted above, $|AO| = |CO|$, and because $\triangle OAB \cong \triangle OCD$, we also have that $\angle OAP = \angle OCQ$. Hence, by the Side-Angle-Side congruence criterion, $\triangle OAP = \triangle OCQ$, and thus $|OP| = |OQ|$. Since the perpendicular from a given point to a line meets the line at the nearest point on that line to the given point (Proposition 2.3.24 in the text, oddly never explicitly stated by Euclid in the *Elements*), it follows that the (nearest!) distances from O to AB and CD are equal. ■

Quiz #9. Friday, 21 March, 2014 *[10 minutes]*

1. Suppose $\triangle ABC$ is equilateral, with sides 4 *shazbats* long. Let E be the point between B and C which is 3 *shazbats* from B, F be the midpoint of AC, and D be the point where EF meets AB.



Determine $|AD|$. *[5]*

Solution. Since $|AB| = |BC| = |CA| = 4$, $|BE| = 3$ implies that $|EC| = 4 - 1 = 3$, and the fact that F is the midpoint of AC implies that $|CF| = |FC| = \frac{4}{2} = 2$. As D, E, and F are on the same line, namely EF, it follows from Menelaus' Theorem that

$$-1 = \frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = \frac{-|AD|}{|DB|} \cdot \frac{|BE|}{|EC|} \cdot \frac{|CF|}{|FA|} = \frac{-|AD|}{|DB|} \cdot \frac{3}{1} \cdot \frac{2}{2} = -3\frac{|AD|}{|DB|},$$

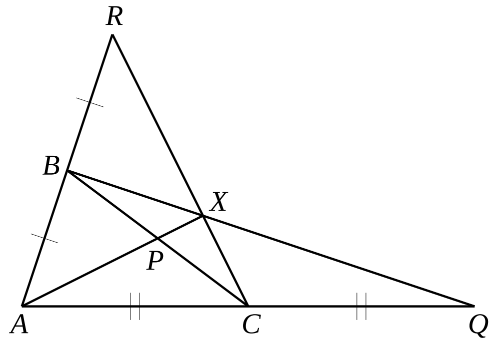
where we choose AB, BC, and CA to be the positive orientations of the sides, so, in particular, $AD = -|AD|$. This means that

$$3|AD| = |DB| = |AD| + |AB| = |AD| + 4 \implies 2|AD| = 4 \implies |AD| = \frac{4}{2} = 2.$$

Thus $|AD| = 2$ *shazbats*. ■

Quiz #10. Friday, 28 March, 2014 *[10 minutes]*

1. Suppose AR is a line segment with midpoint B, and AQ is another line segment with midpoint C, meeting the first line segment at A. Let X be the point of intersection of QB and RC, and let P be the point of intersection of AX and BC.



Show that P is the midpoint of BC. *[5]*

Solution. Consider $\triangle ABC$. R is on (an extension of) side AB, Q is on (an extension of) side AC, and P is on the side BC of this triangle. Note that AP, BQ, and CR are concurrent at X. It follows by Ceva's Theorem that $\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = +1$, using the convention that reversing the direction of a line segment changes its sign. Since B is the midpoint of AR, $AR = -2RB$, and since C is the midpoint of AQ, $QA = -2CQ$. Plugging these into the equation from Ceva's Theorem gives us

$$+1 = \frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{-2RB}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{-2CQ} = (-2) \cdot \frac{BP}{PC} \cdot \left(-\frac{1}{2}\right) = \frac{BP}{PC}.$$

Thus $BP = PC$, so P must be between B and C and equidistant from B and C, *i.e.* P is the midpoint of BC. ■

Quiz #11. Friday, 4 April, 2014 *[10 minutes]*

1. Suppose that the Euler line of $\triangle ABC$ is also the angle bisector of $\angle BAC$. Show that $\triangle ABC$ is isosceles. *[5]*

Solution. If the Euler line of $\triangle ABC$ is also the angle bisector of $\angle BAC$, it will meet BC at some point D between B and C. Since it is an angle bisector, we have that $\angle BAD = \angle CAD$. On the other hand, if it is the Euler line, it passes through the orthocentre of $\triangle ABC$ and, since it also passes through A, it must be the altitude from A. Thus $\angle ADB = \angle ADC$ are right angles. As $|AD| = |AD|$, it follows by the Angle-Side-Angle (ASA) congruence criterion that $\triangle ADB \cong \triangle ADC$, and hence that $|AB| = |AC|$, *i.e.* $\triangle ABC$ is isosceles. ■