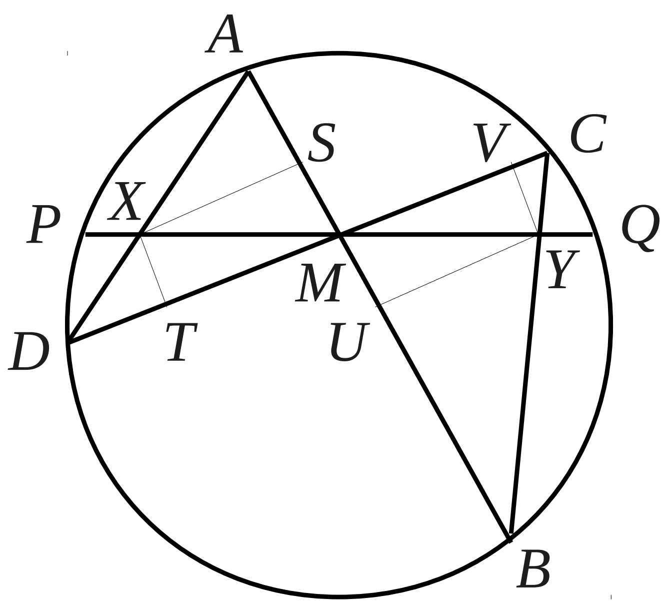
Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2013

Solution to Assignment #9
A circle flutters by ...

The following is a classic result in Euclidean geometry:

The Butterfly Theorem. Suppose M is the midpoint of a chord PQ of a circle and AB and CD are two other chords that pass through M. Let AD and BC intersect PQ at X and Y, respectively. Then M is also the midpoint of XY.



Oops! Forgot to label M ...

1. Prove the Butterfly Theorem. *[10]*

Hint: You know a lot about angles in a circle, and about triangles, and also ...

Solution. Draw perpendiculars from X to AB, meeting AB at S, from X to CD, meeting CD at T, from Y to AB, meeting AB at U, and from Y to CD, meeting CD at V. (See the diagram above!) This yields a wealth of similar triangles, and hence of common ratios of sides:

- $\angle XMS = \angle YMU$ since they are opposite angles, and $\angle XSM = \angle YUM$ since both are right angles, so $\triangle XSM \sim \angle YUM$ by the A-A similarity criterion. It follows in particular that $\frac{|XM|}{|YM|} = \frac{|XS|}{|YU|}$.
- $\angle XMT = \angle YMV$ since they are opposite angles, and $\angle XTM = \angle YVM$ since both are right angles, so $\triangle XTM \sim \angle YVM$ by the A-A similarity criterion. It follows in particular that $\frac{|XM|}{|YM|} = \frac{|XT|}{|YV|}$.

- $\angle XAS = \angle YCV$ since $\angle DAB = \angle XAS$ and $\angle BCD = \angle YCV$ both subtend the same arc of the circle, and $\angle ASX = \angle CVY$ since both are right angles, so $\triangle ASX \sim \angle CVY$ by the A-A similarity criterion. It follows in particular that $\frac{|XS|}{|YV|} = \frac{|AX|}{|CY|}$.
- $\angle XDT = \angle YBU$ since $\angle ADC = \angle XDT$ and $\angle CBA = \angle YBU$ both subtend the same arc of the circle, and $\angle DTX = \angle BUY$ since both are right angles, so $\triangle DTX \sim \angle BUY$ by the A-A similarity criterion. It follows in particular that $\frac{|XT|}{|YU|} = \frac{|DX|}{|BY|}$.

  We will also require the following fact:

  LEMMA. If two chords AB and CD of a circle intersect in a point P, then $|AP| \cdot |PB| = |CP| \cdot |PD|$.

The case where P is outside the circle was the question on Quiz #8. It's to left you to check that it works for a point P inside the circle ...

Here we go, using the various relations noted above:

$$\begin{aligned}\frac{|XM|^2}{|YM|^2} &= \frac{|XM|}{|YM|} \cdot \frac{|XM|}{|YM|} = \frac{|XS|}{|YU|} \cdot \frac{|XT|}{|YV|} = \frac{|XS|}{|YV|} \cdot \frac{|XT|}{|YU|} = \frac{|AX|}{|CY|} \cdot \frac{|DX|}{|BY|} \\ &= \frac{|AX| \cdot |XD|}{|CY| \cdot |YB|} = \frac{|PX| \cdot |XQ|}{|PY| \cdot |YQ|} = \frac{(|PM| - |XM|) \cdot (|QM| + |XM|)}{(|PM| + |YM|) \cdot (|QM| - |YM|)} \\ &= \frac{(|PM| - |XM|) \cdot (|PM| + |XM|)}{(|PM| + |YM|) \cdot (|PM| - |YM|)} = \frac{|PM|^2 - |XM|^2}{|PM|^2 - |YM|^2}\end{aligned}$$

Cross-multiplying the first and last expressions in the chain of equalities gives:

$$|XM|^2|PM|^2 - |XM|^2|YM|^2 = |YM|^2|PM|^2 - |YM|^2|XM|^2$$

Thus $|XM|^2|PM|^2 = |YM|^2|PM|^2$, so $|XM|^2 = |YM|^2$, and hence $|XM| = |YM|$. It follows that M is the midpoint of XY, as desired. ■

NOTE: The proof given above is basically the one given on pp. 45-46 of *Geometry Revisited*, by H.S.M. Coxeter and S.L. Greitzer, *New Mathematical Library* Vol. 19, Random House, 1967. The authors say they received this proof from a "Dr. Zoll of Newark State College." This proof is pretty common nowadays, probably because it is pretty basic in the tools it requires; *e.g.* the Wikipedia article on the Butterfly Theorem gives a version of it.