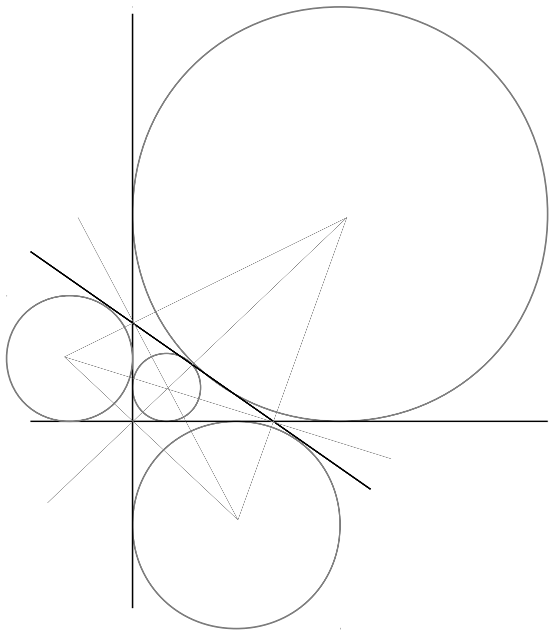
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013
Solution to Assignment #8
Incircle and excircles

1. Show that there four circles which are tangent to (extensions of) all three sides of $\triangle ABC$. *[10]*

Hint: The centres of these circles must be on the bisectors of certain angles.

NOTE. One of these circles fits inside the triangle – this one is called the *incircle* of the triangle and its centre is the triangle's *incentre* – and the other three are outside it – these are the triangles *excircles* and their centres are the triangle's *excentres*.

SOLUTION. Here's an unlabelled sketch of the basic set-up, showing the various angle bisectors involved:



Recall that a point is on the angle bisector of an angle exactly when it is equidistant from the lines defining the angle (*e.g.* Propositions 2.3.32 & 2.3.33 in the textbook). Suppose, then, that $\triangle ABC$ is given.

Let I be the point of intersection of the angle bisectors of $\angle ABC$ and $\angle ACB$. Then I is equidistant from AB and BC because it is on the angle bisector of $\angle ABC$, and it is equidistant from BC and AC because it is on the angle bisector of $\angle ACB$. It follows that I is equidistant from AB and AC, and so I is also on the angle bisector of $\angle BAC$.

Let r be the common distance that I is from each of the sides of $\triangle ABC$ and consider the circle of radius r about I. Since each side of the triangle is a distance of r from I, there is a point on each side of the triangle where the circle touches that side. Since r is the least distance that any point on each side of the triangle is from I, the circle touches each side of the triangles at only one point. It follows that the sides of the triangle are tangent to the circle. This circle is the incircle and its centre I is the incentre of $\triangle ABC$.

Now let E_1 be the point of intersection of the angle bisector of $\angle BAC$ with the angle bisector of the external angle of the triangle at B. (Note that it doesn't matter which external angle at B is meant here, since the same line serves as the angle bisector for both, because the two external angles are opposite angles.) Then E_1 is equidistant from AB and AC (extended as necessary) because it is on the angle bisector of $\angle BAC$, and it is equidistant from AB and BC (extended as necessary) since it is on the bisector of the external angle at B. It follows that E_1 is equidistant from AC and BC, and so is also on an angle bisector of an angle at C. A little reflection – or a glance at the diagram – makes it evident that E_1 is on the angle bisector of the external angle of $\triangle ABC$ at C.

Let r_1 be the common distance that E_1 is from (extensions of) each of the sides of $\triangle ABC$ and consider the circle of radius r_1 about E_1. Since (an extension of) each side of the triangle is a distance of r_1 from E_1, there is a point on (an extension of) each side of the triangle where the circle touches that side. Since r_1 is the least distance that any point on (an extension of) each side of the triangle is from E_1, the circle touches each (extension of a) side of the triangles at only one point. It follows that the (extensions of) sides of the triangle are tangent to the circle. Note that this circle and its centre, E_1, are on the other side of BC from A.

Similar arguments show that there are points E_2 and E_3 on the other sides of the triangle from B and C, respectively, which serve as the centres of circles, with radii r_2 and r_3, respectively, which are also tangent to (extensions of) each side of $\triangle ABC$.

The latter three circles are the excircles and their centres are the excentres of $\triangle ABC$. Because any circle tangent to (extensions of) all three sides of $\triangle ABC$ must be equidistant from them, its centre must be at the angle bisectors intersection of three of the internal or external angles of $\triangle ABC$. Since I, E_1, E_2, and E_3 use up all the possibilities for such intersections – look at the diagram! – the incircle and the three excircles are the only four circles which are tangent to (extensions of) all three sides of the triangle. ■