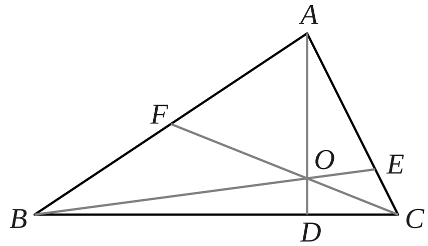
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013

Solution to Assignment #7
Ceva's Theorem

The following result was first proved by the medieval Arab mathematician Yusuf al-Mu'taman ibn Hūd, who was also ruler of Zaragoza from 1082 to 1085. His work was lost for many centuries, though, and in the meantime it was rediscovered and proved again by Giovanni Ceva (1647-1734), an Italian Jesuit mathematician.

CEVA'S THEOREM. Suppose D, E, and F are points on the sides BC, AC, and AB, respectively, of $\triangle ABC$. Then AD, BE, and CF all meet in a single point O if and only if $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$.



The statement above is not quite the most general: with some additional definitions, there is a version that also works when two of D, E, and F are on extensions of the sides and O is outside the triangle. However, the limited version given above is still useful.

1. Prove half of Ceva's Theorem: Suppose D, E, and F are points on the sides BC, AC, and AB, respectively, of $\triangle ABC$. Then if AD, BE, and CF all meet in a single point O, then $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = 1$. *[10]*

Hint: Rewrite $\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|}$ in terms of the areas of certain triangles in the picture.

SOLUTION. Observe that $\triangle ACF$ and $\triangle FCB$ have the same height, so their areas are proportional to their bases, which are AF and FB, respectively. hence $\frac{|AF|}{|FB|} = \frac{\text{area}(\triangle ACF)}{\text{area}(\triangle FCB)}$. Since $\triangle AOF$ and $\triangle FOB$ also have the same height, their areas are also proportional to their bases, namely AF and FB, respectively, and hence we also have $\frac{|AF|}{|FB|} = \frac{\text{area}(\triangle AOF)}{\text{area}(\triangle FOB)}$. With little algebra, it now follows that

$$\begin{aligned}\frac{|AF|}{|FB|} &= \frac{\text{area}(\triangle ACF)}{\text{area}(\triangle FCB)} = \frac{\text{area}(\triangle AOF)}{\text{area}(\triangle FOB)} \\ &= \frac{\text{area}(\triangle ACF) - \text{area}(\triangle AOF)}{\text{area}(\triangle AOF) - \text{area}(\triangle FOB)} = \frac{\text{area}(\triangle AOC)}{\text{area}(\triangle BOC)}\,.\end{aligned}$$

Similar arguments show that $\frac{|BD|}{|DC|} = \frac{\text{area}(\triangle AOB)}{\text{area}(\triangle AOC)}$ and $\frac{|CE|}{|EA|} = \frac{\text{area}(\triangle BOC)}{\text{area}(\triangle AOB)}$. It then follows that

$$\frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = \frac{\text{area}(\triangle AOC)}{\text{area}(\triangle BOC)} \cdot \frac{\text{area}(\triangle AOB)}{\text{area}(\triangle AOC)} \cdot \frac{\text{area}(\triangle BOC)}{\text{area}(\triangle AOB)} = 1\,,$$

as desired. ■