Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2013

Solutions to Assignment #6
Circumcircles

A circle that passes through all three vertices of $\triangle ABC$ is called the *circumcircle* of the triangle and its centre is the triangle's *circumcentre*.

1. Show that the perpendicular bisectors of the sides of $\triangle ABC$ meet in a point that is equidistant from all three vertices. *[6]*

Solution. We'll first prove a lemma (mentioned in class and given in the textbook) that is related to Proposition I-10. (You were welcome to use it without proving it ...)

> Lemma. A point C not on AB is on the perpendicular bisector to AB if and only if C is equidistant from A and B, *i.e.* $|AC| = |BC|$.
>
> Proof. ($\Longrightarrow$) Suppose D is the point on AB where the perpendicular bisector intersects it and C is any other point on the perpendicular bisector. This means that $\angle ADC = \angle BDC$ are right angles and D is the midpoint of AB, *i.e.* $|AD| = |BD|$. Since $|CD| = |CD|$, it follows by the Side-Angle-Side (SAS) congruence criterion that $\triangle ADC \cong \triangle BDC$. Hence $|AC| = |BC|$.
>
> ($\Longleftarrow$) Suppose C is not on AB and $|AC| = |BC|$. Let D be the midpoint of AB. Then $|AD| = |BD|$ and $|CD| = |CD|$ as well, so $\triangle ADC \cong \triangle BDC$ by the Side-Side-Side (SSS) congruence criterion. It follows that $\angle ADC = \angle BDC$ and, since the angles add up to the straight angle $\angle ADB$, it follows that $\angle ADC = \angle BDC$ are right angles. since we also have that $|AD| = |BD|$, it follows that CD is (part of) the perpendicular bisector of AB. $\square$

Suppose then that $\triangle ABC$ is given. Let O be the point of intersection of the perpendicular bisectors of AB and BC. Then, by the Lemma above, $|AO| = |BO|$ and $|BO| = |CO|$. It follows that $|BO| = |CO|$ too, so by the Lemma O is also on the perpendicular bisector of AC. $\blacksquare$

2. Use **1** to help show that any triangle has an unique circumcircle. *[4]*

Solution. Suppose $\triangle ABC$ is given. Let O be the point where the perpendicular bisectors of the sides of $\triangle ABC$ meet (per **1** above). Since $|AO| = |BO| = |CO|$, the circle with centre O and radius $|AO|$ passes through all three of A, B, and C, so it is a circumcircle of the triangle.

It remains to show that the circumcircle of the triangle is unique. Suppose a circle with centre P passes through all three of A, B, and C. Then $|PA| = |PB| = |PC|$ since all three line segments are radii of the circle. It follows by the Lemma that P is on all three of the perpendicular bisectors of the sides of the triangle, that is $P = O$, and so also $|AO| = |PO|$. Hence the circle is the same circle that was obtained above, *i.e.* the circumcircle is unique. $\blacksquare$