Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013

Solutions to Assignment #5
Triggy business with similarity

Recall from the previous assignment that $\triangle ABC$ and $\triangle DEF$ are *similar*, denoted by $\triangle ABC \sim \triangle DEF$, if the ratios of the lengths of corresponding sides are all the same, that is, if $\dfrac{|AB|}{|DE|} = \dfrac{|BC|}{|EF|} = \dfrac{|AC|}{|DF|}$.

Two handy facts about triangles, if one has trigonometry in hand, are:

THE LAW OF SINES: In any triangle $\triangle ABC$,

$$\frac{\sin(\angle BAC)}{|BC|} = \frac{\sin(\angle ACB)}{|AB|} = \frac{\sin(\angle ABC)}{|AC|}\,.$$

and

THE LAW OF COSINES: In any triangle $\triangle ABC$,

$$|AC|^2 = |AB|^2 + |BC|^2 - 2\cdot|AB|\cdot|BC|\cdot\cos(\angle ABC)\,.$$

You may use these laws, and whatever else you know or learn about the trigonometric functions, to help do the following problems.

1. Show that if $\angle BAC = \angle EDF$ and $\dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|}$, then $\triangle ABC \sim \triangle DEF$. *[4]*

NOTE: That is, show that a Side-Angle-Side (SAS) similarity criterion for triangles works.

SOLUTION. We will use the Law of Cosines twice to help:

$$\begin{aligned}
|BC|^2 &= |AB|^2 + |AC|^2 - 2\cdot|AB|\cdot|AC|\cdot\cos(\angle ABC) \\
&\qquad \text{(By the Law of Cosines.)} \\
&= \left[|AC|\cdot\frac{|DE|}{|DF|}\right]^2 + |AC|^2 - 2\cdot\left[|AC|\cdot\frac{|DE|}{|DF|}\right]\cdot|AC|\cdot\cos(\angle EDF) \\
&\qquad \text{(Since } \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} \Longrightarrow |AB| = |AC|\cdot\frac{|DE|}{|DF|} \text{ and } \angle BAC = \angle EDF.\text{)} \\
&= |AC|^2\cdot\frac{|DE|^2}{|DF|^2} + |AC|^2 - 2\cdot|AC|^2\cdot\frac{|DE|}{|DF|}\cdot\cos(\angle EDF) \\
&= \frac{|AC|^2}{|DF|^2}\cdot\left[|DE|^2 + |DF|^2 - 2\cdot|DE|\cdot|DF|\cdot\cos(\angle EDF)\right] \\
&= \frac{|AC|^2}{|DF|^2}\cdot|EF|^2 \qquad \text{(By the Law of Cosines.)}
\end{aligned}$$

It follows that $\dfrac{|BC|^2}{|EF|^2} = \dfrac{|AC|^2}{|DF|^2}$; taking the square roots of both sides gives $\dfrac{|AC|}{|DF|} = \dfrac{|BC|}{|EF|}$. Thus $\dfrac{|AB|}{|DE|} = \dfrac{|AC|}{|DF|} = \dfrac{|BC|}{|EF|}$, so $\triangle ABC \sim \triangle DEF$ by definition. ■

2. State and prove a Side-Side-Side (SSS) criterion for similarity. *[2]*

Solution. Trick question! The obvious Side-Side-Side (SSS) criterion for similarity is to require that $\dfrac{|AB|}{|DE|} = \dfrac{|BC|}{|EF|} = \dfrac{|AC|}{|DF|}$. This is our *definition* of similarity for triangles, so there isn't really anything to prove ... ■

3. If $\triangle ABC \sim \triangle DEF$, then $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. *[4]*

Note: That is, show that in similar triangles corresponding angles are equal.

Solution. $\triangle ABC \sim \triangle DEF$, so $\dfrac{|AB|}{|DE|} = \dfrac{|BC|}{|EF|} = \dfrac{|AC|}{|DF|}$ by definition. Note that with a little algebra it follows that $\dfrac{|AB|}{|BC|} = \dfrac{|DE|}{|EF|}$, $\dfrac{|AC|}{|BC|} = \dfrac{|DF|}{|EF|}$, and $\dfrac{|AB|}{|AC|} = \dfrac{|DE|}{|DF|}$. Rearranging the Law of Cosines to solve for the angle and exploiting the relationships just obtained gives:

$$\begin{aligned}
\cos(\angle ABC) &= \frac{|AC|^2 - |AB|^2 - |BC|^2}{2 \cdot |AB| \cdot |BC|} = \frac{1}{2}\left(\frac{|AC|}{|AB|} \cdot \frac{|AC|}{|BC|} - \frac{|AB|}{|BC|} - \frac{|BC|}{|AB|}\right) \\
&= \frac{1}{2}\left(\frac{|DF|}{|DE|} \cdot \frac{|DF|}{|EF|} - \frac{|DE|}{|EF|} - \frac{|EF|}{|DE|}\right) = \frac{|DF|^2 - |DE|^2 - |EF|^2}{2 \cdot |DE| \cdot |EF|} \\
&= \cos(\angle DEF)
\end{aligned}$$

Since cos is a $1-1$ function for angles between 0 and a straight angle – which the internal angles of a triangle must be (Why?) – it follows that $\angle ABC = \angle DEF$.

Similar calculations will show that $\angle BCA = \angle EFD$ and $\angle CAB = \angle FDE$. ■

□. This is unquestionably a non-question! *[0]*

Non-solution. What else for a non-question? ▼