Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013

Solutions on Assignment #4
Congruence and Similarity

1. Show that if $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $|CA| = |FD|$, then $\triangle ABC \cong \triangle DEF$. *[5]*

NOTE: That is, show that the Angle-Angle-Side (AAS) congruence criterion for triangles works.

SOLUTION. Suppose $\triangle ABC$ and $\triangle DEF$ have $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $|CA| = |FD|$. Apply $\triangle ABC$ to $\triangle DEF$ so that A is on D, AC lies along DF, and B is on the same side of DF as E.

Since A is on D, AC lies along DF, and $|AC| = |DF|$, it follows that C is on F. Also, because $\angle BCA = \angle EFD$, it also follows that AB lies along DE. There are now appear three possibilities:

i. *B is on E.* In this case, all three vertices of $\triangle ABC$ are on the corresponding vertices of $\triangle DEF$, so $\triangle ABC \cong \triangle DEF$, as desired.

ii. *B is on DE strictly between D and E.* In this case, $\angle ABC = \angle DBF$ is an exterior angle and $\angle BEF = \angle DEF$ is an opposite interior angle of $\triangle BEF$. It follows that $\angle ABC > \angle DEF$, contradictiong the given fact that $\angle ABC = \angle DEF$.

iii. *B is on DE beyond E.* In this case, $\angle DEF$ is an exterior angle and $\angle DBF = \angle ABC$ is an opposite interior angle of $\triangle BEF$. It follows that $\angle ABC < \angle DEF$, contradictiong the given fact that $\angle ABC = \angle DEF$.

Thus both cases *ii* and *iii* lead to a contradiction, leaving case *i*, in which the desired conclusion holds, as the only possibility. ■

DEFINITION. $\triangle ABC$ and $\triangle DEF$ are *similar*, often denoted by $\triangle ABC \sim \triangle DEF$, if the ratios of (the lengths of) corresponding sides are all the same, *i.e.* if $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$.

2. Show that congruence implies similarity for triangles, *i.e.* $\triangle ABC \cong \triangle DEF$ implies that $\triangle ABC \sim \triangle DEF$. Give an example to show that the converse is not necessarily true. *[1]*

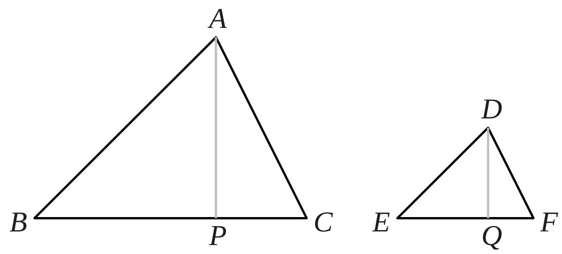
SOLUTION. If $\triangle ABC \cong \triangle DEF$, then $|AB| = |DE|$, $|BC| = |EF|$, and $|AC| = |DF|$, so $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = 1$. Thus $\triangle ABC \sim \triangle DEF$. ■

3. Use trigonometry to show that if $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$, then $\triangle ABC \sim \triangle DEF$.

NOTE: That is, show that the Angle-Angle (AA) criterion for similarity works. Besides trigonometry, you may assume – as trigonometry in Euclidean space does – that the sum of the interior angles of a triangle is $180° = \pi\ rad =$ a straight angle.

We will revisit the definition and basics of similarity once we start using the parallel axiom (which similarity really needs), but introducing the concept now will allow us to investigate some uses of it before that.

SOLUTION. Suppose $\triangle ABC$ and $\triangle DEF$ with $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$ are given. Let P and Q be the points on BC and EF, respectively, where the altitudes from A and D, respectively, meet the opposite side of the triangle. (So $AP \perp BC$ and $DQ \perp EF$.)



$\triangle ABP$, $\triangle ACP$, $\triangle DEQ$, and $\triangle DFQ$ are all right triangles. From the properties of the basic trigonometric functions and what we are given about the angles of the triangles it now follows that

$$\begin{aligned}
&\frac{|PB|}{|AB|} = \cos(\angle ABC) = \cos(\angle DEF) = \frac{|QE|}{|DE|} \implies \frac{|AB|}{|DE|} = \frac{|PB|}{|QE|}\,,\\
&\frac{|PC|}{|AC|} = \cos(\angle BCA) = \cos(\angle EFD) = \frac{|QF|}{|DF|} \implies \frac{|AC|}{|DF|} = \frac{|PC|}{|QF|}\,,\\
&\frac{|AP|}{|AB|} = \sin(\angle ABC) = \sin(\angle DEF) = \frac{|DQ|}{|DE|} \implies \frac{|AB|}{|DE|} = \frac{|AP|}{|DQ|}\,,\\
\text{and}\quad &\frac{|AP|}{|AC|} = \sin(\angle BCA) = \sin(\angle EFD) = \frac{|DQ|}{|DF|} \implies \frac{|AC|}{|DF|} = \frac{|AP|}{|DQ|}\,.
\end{aligned}$$

Thus

$$\frac{|AP|}{|DQ|} = \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|PB|}{|QE|} = \frac{|PC|}{|QF|}\,.$$

From $\frac{|PB|}{|QE|} = \frac{|PC|}{|QF|}$ it follows that $\frac{|PB|}{|QE|} = \frac{|PC|}{|QF|} = \frac{|PB|+|PC|}{|QE|+|QF|} = \frac{|BC|}{|EF|}$ – you figure out why! – and thus we have

$$\frac{|AB|}{|DE|} = \frac{|AC|}{|DF|} = \frac{|BC|}{|EF|}\,,$$

so $\triangle ABC \sim \triangle DEF$, as desired. ■