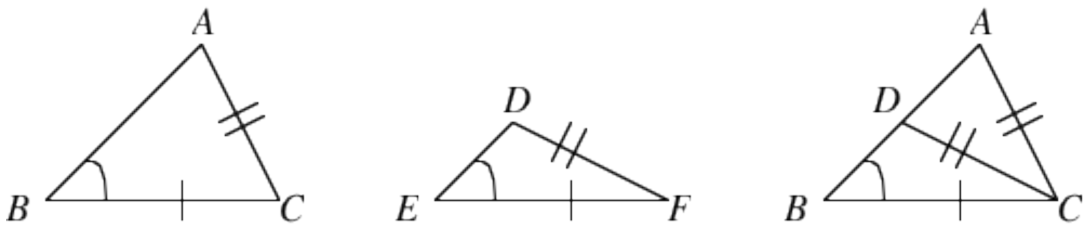
Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2013

Solutions to Assignment #3
Congruence Criteria

1. Find triangles $\triangle ABC$ and $\triangle DEF$ such that $\angle ABC = \angle DEF$, $|BC| = |EF|$, and $|CA| = |FD|$, but $\triangle ABC \ncong \triangle DEF$ (*i.e.* $\triangle ABC$ is not congruent to $\triangle DEF$). *[5]*

Note: Such an example shows that the Angle-Side-Side (ASS) congruence criterion for triangles does not work.

Solution. If $\angle ABC = \angle DEF$ is acute, it is possible to construct a counterexample:



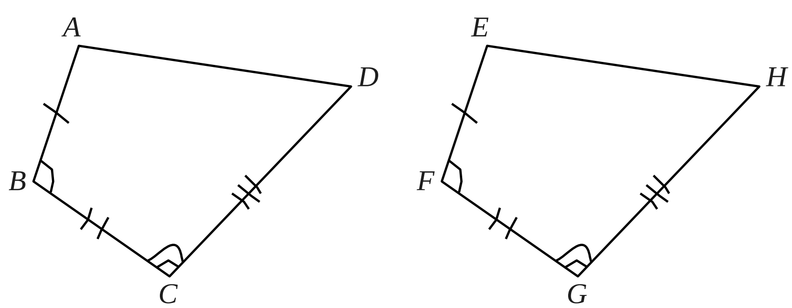
This example was obtained by starting with an isosceles triangle $\triangle ACD$, extending AD past D to some point B, and then letting $B = E$ and $C = F$.

By the way, if $\angle ABC = \angle DEF$ is a right or an obtuse angle, this trick doesn't work. In those cases, Angle-Side-Side congruence does work. ■

2. Show that if quadrilaterals $\square ABCD$ and $\square EFGH$ (neither of which has sides crossing except at the vertices) satisfy $|AB| = |EF|$, $\angle ABC = \angle EFG$, $|BC| = |FG|$, $\angle BCD = \angle FGH$, and $|CD| = |GH|$, then $\square ABCD \cong \square EFGH$. *[5]*

Note: That is, you need to show that the Side-Angle-Side-Angle-Side (SASAS) congruence criterion for quadrilaterals does work.

Solution. One can do this directly with (an obvious extension of) the Application Postulate or by dividing the quadrilaterals into triangles and applying various congruence criteria to the triangles. Just for fun, we'll do both.



Using Application. Step by step, we indulge in overkill:

0. Apply $\Box ABCD$ to $\Box EFGH$, placing A on E, AB along EF, and C and D on the same side(s) of EF as G and H, respectively.
1. Since $|AB| = |EF|$, A is on E, and AB is along EF, B must be on F.
2. Since B is on F, $\angle ABC = \angle EFG$, and C and G are on the same side of EF, BC lies along FG.
3. Since $|BC| = |FG|$, B is on F, and BC lies along FG, C must be on G.
4. Since C is on G, $\angle BCD = \angle FGH$, and D and H are on the same side of EF, CD lies along GH.
5. Since $|CD| = |GH|$, C is on G, and CD lies along GH, D must be on H.
6. Since A is on E and D is on H, $|AD| = |EH|$.
7. Since B is on F, A is on E, and D is on H, $\angle BAD = \angle FEH$.
8. Since A is on E, D is on H, and C is on G, $\angle ADC = \angle EHG$.
9. Since all corresponding side-lengths and angles have now been given or shown to be equal, $\Box ABCD \cong \Box EFGH$.

   Whew! (There is actually one step where the argument given isn't quite enough, if one is very strict. What is it?) $\Box$

Divide into triangles and and conquer. More overkill:

. Connect A to C and E to G.

i. Since $|AB| = |EF|$, $\angle ABC = \angle EFG$, and $|BC| = |FG|$, we have $\triangle ABC \cong \triangle EFG$ by the SAS criterion.

ii. Since $\triangle ABC \cong \triangle EFG$, $|AC| = |EG|$, $\angle BAC = \angle FEG$, and $\angle BCA = \angle FGE$.

iii. Since $\angle BCD = \angle FGH$ and $\angle BCA = \angle FGE$, we have $\angle ACD = \angle BCD - \angle BCA = \angle FGH - \angle FGE = \angle EGH$.

iv. Since $|AC| = |EG|$, $\angle ACD = \angle EGH$, and $|CD| = |GH|$, we have $\triangle ACD \cong \triangle EGH$.

v. Since $\triangle ACD \cong \triangle EGH$, $|AD| = |EH|$, $\angle CDA = \angle GHE$, and $\angle CAD = \angle GEH$.

vi. Since $\angle BAC = \angle FEG$ and $\angle CAD = \angle GEH$, we have that $\angle BAD = \angle BAC + \angle CAD = \angle FEG + \angle GEH = \angle FEH$.

vii. Since all corresponding side-lengths and angles have now been given or shown to be equal, $\Box ABCD \cong \Box EFGH$.

Whew, again! $\blacksquare$