Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013

Solution to Assignment #2
Six sides, no waiting ...

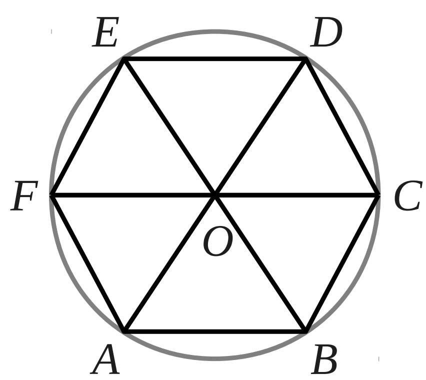
A *regular hexagon* is a convex polygon with six sides, each of which is a straight line, of the same length, such that each interior angle (where two sides meet) is equal to every other interior angle of the polygon.

1. Given a straight line, use Postulates I-IV, S (Separation), and/or A (Application), to show there is a regular hexagon with the given straight line as one of its sides. *[3]*

NOTE: You may take Euclid's Proposition I-1 for granted, too.

SOLUTION. Suppose we are given a straight line AB. we will construct a regular heaxagon on AB as follows:

1. Use Proposition I-1 to construct an equilateral triangle $\triangle ABO$ on AB.
2. Use Proposition I-1 to construct an equilateral triangle $\triangle OBC$ on OB.
3. Use Proposition I-1 to construct an equilateral triangle $\triangle OCD$ on OC.
4. Use Proposition I-1 to construct an equilateral triangle $\triangle ODE$ on OD.
5. Use Proposition I-1 to construct an equilateral triangle $\triangle OEF$ on OE.
6. Draw the line segment FA using Postulate I.



We need to verify that the hexagon $ABCDEF$ is indeed regular.

The equilateral triangles constructed in steps 1–5 above all have the same side length, namely $|AB|$: at each step we used AB or the side of an equilateral triangle of side length $|AB|$ to construct the next equilateral triangle. It follows that $|AB| = |BC| = |CD| = |DE| = |EF|$,

Since the five equilateral triangles constructed in steps 1–5 above all have the same side length for all three sides, they are all congruent by the Side-Side-Side (SSS) congruence criterion. It follows that all of their interior angles are equal, and so $\angle ABC = \angle BCD = \angle CDE = \angle DEF$, since each of these interior angles of the hexagon is the sum of two interior angles of adjacent equlateral triangles.

It still remains to show that $|FA| = |AB|$ and that $\angle EFA$ and $\angle FAB$ are equal to the other interior angles of the hexagon. It is enough to show, in light of the arguments used in the last two paragraphs, that $\triangle OAF$ is also an equilateral triangle of side length $|AB|$.

We already know from the construction of the various equilateral triangles in steps 1–5 above that $|AB| = |OA| = |OF|$. Unfortunately, showing that $\angle AOF$ is the same as the interior angles of the known equilateral triangles, so that we could apply the SAS criterion, or showing that $|FA| = |OA| = |OF|$, so that we could use the SSS criterion, takes some work. In fact, it requires the use of the parallel axiom in some form; the assertion that the sum of the interior angles of a triangle is equal to a straight angle is probably the easiest to use here. Details left to the reader!

In light of the above, you might ask yourself if it actually possible to construct a regular hexagon on any given line segment using only Postulates I–IV. [Presumably by some different method …] *Up to five [5] assignment bonus points to anyone who can answer this, with proof, one way or the other, on or before the end of classes this term!* ■