Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2013

Solutions to Assignment #1
A geometry on a cylinder
Look, Ma! I did it without any pictures!

We will define a geometry $\mathcal{G}$ of points and *lines* on (the surface of) the cylinder $x^2 + y^2 = 1$.

The points of the geometry $\mathcal{G}$ are the points of the cylinder, *i.e.* the points (x, y, z) in three-dimensional Cartesian coordinates which satisfy the equation $x^2 + y^2 = 1$. The *lines* of the geometry $\mathcal{G}$ are the curves on the cylinder obtained by intersecting a plane through the origin in three-dimensional space with the cylinder. (Note that every plane through the origin does indeed intersect the cylinder.) Your task will be to determine some of the basic properties of $\mathcal{G}$.

1. Suppose P and Q are two distinct points of $\mathcal{G}$. Show that there is at least one *line* ℓ of $\mathcal{G}$ which passes through both P and Q. Is such a line necessarily unique? Explain why or why not. *[3]*

SOLUTION. If P and Q (considered as points in three-dimensional space) are not in a straight line with the origin $O = (0, 0, 0)$, then O, P, and Q determine a unique plane in three dinensional space; the intersection of this plane with the cylinder is then the unique *line* of $\mathcal{G}$ passing through both P and Q.

On the other hand, if P and Q are in straight line with the origin O, then there are infinitely many planes passing through all three points; their intersections with the cylinder give infinitely many *lines* of $\mathcal{G}$ passing through both P and Q.

Either way, there is at least one *line* which passes through both P and Q. The case where P and Q are in straight line with the origin O demonstrates that the *line* is not always unique. ■

2. Give (different! :-) examples to show that it is possible for distinct *lines* ℓ and m of $\mathcal{G}$ to intersect in 0 or 2 points of $\mathcal{G}$. *[3]*

SOLUTION. Consider the *lines* a, b, and c of $\mathcal{G}$ obtained by intersecting, respectively, the planes $x = 0$, $y = 0$, and $z = 0$ with the given cylinder.

Lines a and b intersect in 0 points of $\mathcal{G}$ because the planes $x = 0$ and $y = 0$ intersect only in the z-axis, which does not meet the surface of the given cylinder.

On the other hand, *lines* a and c meet in 2 points of $\mathcal{G}$ because the planes $x = 0$ and $z = 0$ meet the cylinder $x^2 + y^2 = 1$ in the points $(0, 1, 0)$ and $(0, -1, 0)$. ■

3. Explain why distinct *lines* ℓ and m of $\mathcal{G}$ cannot intersect in just 1 point of $\mathcal{G}$. *[1]*

SOLUTION. Suppose *lines* ℓ and m of $\mathcal{G}$ intersect in a point P of $\mathcal{G}$. This means that the underlying planes, which by definition must pass through the origin O, must also pass through P. This means that the two planes must intersect in the straight line of three-dimensional space joining O and P. However, this straight line – and hence both planes – must intersect the cylinder again on the other side of O from P; this second point is another point besides P of $\mathcal{G}$ which *lines* ℓ and m intersect. ■

4. Determine whether or not *Playfair's Axiom*,

> If P is a point not on the *line* ℓ, then there is a unique *line* m through P such that l and m do not intersect in any point.

is true in $\mathcal{G}$. *[3]*

SOLUTION. Playfair's Axiom is not true in $\mathcal{G}$. Consider the point $P = (1, 0, 1)$ and the line ℓ of $\mathcal{G}$ obtained by intersecting the plane $z = 0$ with the cylinder $x^2 + y^2 = 1$. Note that P is not on ℓ.

Suppose m is any *line* of $\mathcal{G}$ passing through P. The plane whose intersection with the cylinder is m cannot be horizontal, since it must pass through both $O = (0, 0, 0)$ and $P = (1, 0, 1)$. It follows that it must intersect the plane $z = 0$ in some straight line through O which is not vertical. The points in which this straight line meets the cylinder (which must exist because?) are points of $\mathcal{G}$ which both ℓ and m pass through. Thus Playfair's Axiom fails in $\mathcal{G}$ in this case.

In fact, Playfair's Axiom fails almost all of the time in $\mathcal{G}$. What are the exceptions like? ■

Bonus. Suppose a right-angled triangle on the surface of a sphere of radius R has sides of lengths a and b and a hypotenuse of length c respectively. (Note that the sides of the triangle are all pieces of great circles of the sphere.) Show that $\cos(a/R) \cdot \cos(b/R) = \cos(c/R)$. *[1]*

NOTE: I learned this formula from a short story, *The New Physics: The Speed of Lightness, Curved Space, and Other Heresies*, by Charles Sheffield. It first appeared in ANALOG, Vol. 100, No. 9, September 1980, and was reprinted in the anthology *Hidden Variables,* by Charles Sheffield, Ace Science Fiction, New York, 1981. It's a wonderful spoof of Galileo, Einstein, and physics in general. (All the formulas in it are actually correct in context, I think . . . :-)

SOLUTION. You're on your own! (I may want to use this problem as a bonus in some future year . . . :-) ■