Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2013

Solutions to the Quizzes

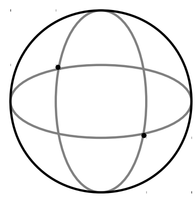
Quiz 1. Friday, 18 January, 2013. *(10 minutes)*

1. Suppose we are given a sphere of radius R and three great circles are drawn on this sphere. What are the possibilities for how many regions these three great circles could subdivide the (surface of the) sphere into? Explain why. *[5]*

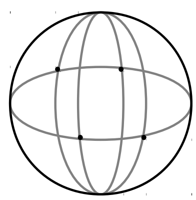
Note: "Regions" are contiguous areas whose borders are pieces of those great circles and which do not have another great circle passing through them. Informally, what the question is really asking is: if we cut an orange through its centre (all the way through the orange) three times, how many pieces of rind might we have at the end?

Solution. There are two possibilities, six (6) regions and eight (8) regions, depending on whether the three great circles all pass through a common point or not, respectively.

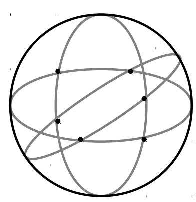
First, recall that two great circles must intersect in exactly two points, which are antipodal (that is, exactly opposite each other on the sphere). It is easy to see that two great circles divide up the sphere into four regions.



Second, if we add a third great circle that passes through one of the points where the first two meet, it also passes through the other point where the first two meet as well. (Why?) It is not hard to see that in this case the new great circle subdivides each of two of the regions made by the first two great circles into two smaller regions. This means we have four of the new, subdivided, regions, plus the two old regions, for a total of six (6) regions.



Finally, if we add a third great circle that does not pass through either point where the first two meet, it subdivides every one of the four regions made by the first two great circles into two smaller regions, for a total of eight (8) regions.

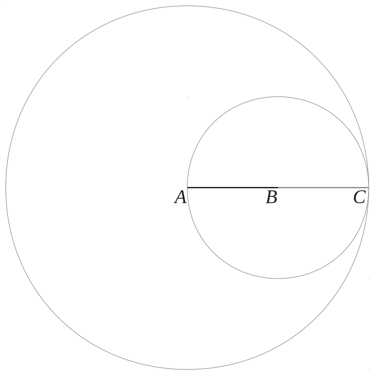


Thar's all, folks! ■

Quiz 2. Friday, 25 January, 2013. *(10 minutes)*

1. Given a line segment AB, use Postulates I–IV, A, and/or S to show that there is a circle with centre A whose radius is twice the length of AB. *[5]*

SOLUTION. Draw a circle with centre B and radius BA [Postulate III]. Extend AB past B until it meets the circle just drawn at C [Postulates II and S]. Finally, draw a circle with centre A and radius AC [Postulate III].



$|BC| = |AB|$ since both are radii of the first circle [definition of a circle], and hence $|AC| = |AB| + |BC| = 2|AB|$, so the second circle is what was asked for. ■

Quiz 3. Friday, 1 February, 2013. *(10 minutes)*

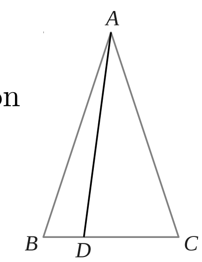
1. Assume $\triangle ABC$ is equilateral and that D is the mid-point of BC. Show that $\angle BAD = \angle CAD$ (so AD is the angle-bisector of $\angle BAC$) and that $\angle ADB$ is a right angle. *[5]*

SOLUTION. We have $|AB| = |AC|$ since $\triangle ABC$ is equilateral, $|AD| = |AD|$ since any thing is equal to itself, and $|BD| = |CD|$ because we are given that D is the mid-point of BC. It follows by the Side-Side-Side (SSS) congruence criterion that $\triangle BAD \cong \triangle CAD$.

Since $\triangle BAD \cong \triangle CAD$, $\angle BAD = \angle CAD$ and $\angle ADB = \angle CDB$. Since $\angle ADB + \angle CDB = \angle BDC =$ a straight angle, it follows that $\angle ADB$ (and $\angle CDB$ too!) is a straight angle. ■

Quiz 4. Friday, 8 February, 2013. *(10 minutes)*

1. Suppose $\triangle ABC$ is isosceles with $|AB| = |AC|$ and D is any point on BC strictly between B and C. Show that $|AD| < |AB|$. *[5]*

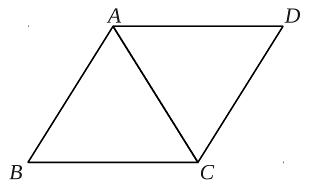


SOLUTION. Since $\triangle ABC$ is isosceles with $|AB| = |AC|$, $\angle ABC = \angle ACB$ [I-5]. Observe that $\angle ADB$ is an exterior angle of $\triangle ACD$, so it is greater than the opposite interior angle $\angle ACD = \angle ACB = \angle ABC$ [I-16], *i.e.* $\angle ADB > \angle ABC = \angle ABD$. Since the greater angle subtends the longer side in $\triangle ADB$ [I-19], it follows that $|AB| > |AD|$, as desired. ■

Quiz 5. Friday, 15 February, 2013. *(10 minutes)*

1. Without using Postulate V (or any equivalent), show that there exists a parallellogram, *i.e.* a quadrilateral $\square ABCD$ such that $AB \parallel CD$ and $AD \parallel BC$. *[5]*

Solution. Let AB be any line segment. Using Proposition I-1, we can find a point C such that $\triangle ABC$ is equilateral; using proposition I-1 again, we can find another point D, on the opposite side of AC from B, such that $\triangle ACD$ is equilateral too.



Note that since $\triangle ABC$ is equilateral, $|AB| = |BC| = |AC|$, and since $\triangle ACD$ is equilateral, $|AC| = |CD| = |AD|$. Since the two triangles have side AC in common, it follows that $|AB| = |BC| = |AC| = |CD| = |AD|$, so $\triangle ABC \cong \triangle CDA$. In turn, this means that $\angle ACB = \angle CAD$, so, by Proposition I-27 – which does not require Postulate V, $AD \parallel BC$. Similarly, $\angle BAC = \angle DCA$, so, by Proposition I-27, $AB \parallel CD$. Thus $\square ABCD$ is a parallelogram. ■

Quiz 6. Friday, 1 March, 2013. *(10 minutes)*

1. Suppose three squares, $\square ABCD$, $\square EFGH$, and $\square IJKL$, are given. Show that there is a square $\square MNOP$ whose area is equal to the sum of the areas of the three given squares. *[5]*

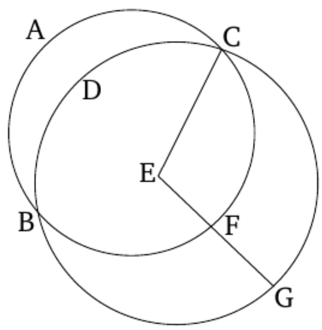
Solution. Draw QR such that $|QR| = |AB|$, and draw QS such that $QS \perp QR$ and $|QS| = |EF|$. Then $\triangle QRS$ is a right triangle and so, by the Pythagorean Theorem, a square with sides of length $|SR|$ is equal in area to the sum of of the areas of $\square ABCD$ and $\square EFGH$. Such a square can be explicitly constructed – if you really care – on the side RS of $\triangle QRS$ by Proposition I-46.

Now draw TU such that $|TU| = |SR|$, and draw TV such that $TV \perp TU$ and $|TV| = |IJ|$. then $\triangle TUV$ is a right triangle and so, by the Pythagorean Theorem, a square $\square MNOP$ with sides of length $|UV|$ must have area equal to the sum of the areas of $\square ABCD$, $\square EFGH$, and $\square IJKL$. Such a square can be explicitly constructed – if you really care – on the side UV of $\triangle TUV$ by Proposition I-46. ■

Quiz 7. Friday, 8 March, 2013. *(10 minutes)*

1. Suppose two circles intersect at (and only at!) two different points. Show that they do not have the same centre. *[5]*

SOLUTION. This problem is actually a slight restatement of Proposition III-5 in Euclid's *Elements.* Here is Euclid's proof:



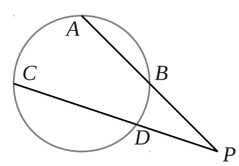
For let the two circles ABC and CDG cut one another at points B and C. I say that they will not have the same center.

For, if possible, let E be (the common center), and let EC have been joined, and let EFG have been drawn through (the two circles), at random. And since point E is the center of the circle ABC, EC is equal to EF. Again, since point E is the center of the circle CDG, EC is equal to EG. But EC was also shown (to be) equal to EF. Thus, EF is also equal to EG, the lesser to the greater. The very thing is impossible. Thus, point E is not the (common) center of the circles ABC and CDG.

Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show. ■

Quiz 8. Friday, 15 March, 2013. *(10 minutes)*

1. Suppose that the extensions of chords AB and CD of a circle intersect in a point P outside the circle. Show that $|PA| \cdot |PB| = |PC| \cdot |PD|$. *[5]*



SOLUTION. This problem is actually a slight restatement of Proposition III-35 in Euclid's *Elements.* Since we proved Propositions III-36 and 37 in class without reference to III-35, we can use them to give a proof of III-35 that is different from Euclid's:

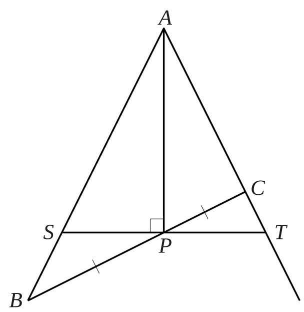
Draw a line PT such that PT is tangent to the circle at T. By III-36 and 37, $|PA| \cdot |PB| = |PT|^2$ and $|PC| \cdot |PD| = |PT|^2$, and so $|PA| \cdot |PB| = |PC| \cdot |PD|$. ■

Quiz 9. Friday, 22 March, 2013. *(12 minutes)*

1. Suppose that in $\triangle ABC$, P is the midpoint of BC and Q and R are points on AC and AB, respectively, so that BQ and CR are the angle-bisectors of $\angle ABC$ and $\angle ACB$, respectively. Show that if AP, BQ, and CR are concurrent in a point X, then $\triangle ABC$ is isosceles. *[5]*

SOLUTION. Since X is the intersection of two of the internal angle bisectors of $\triangle ABC$, it is actually the incentre of the triangle, *i.e.* $X = I$. Since all three angle bisectors meet at the incentre, this means that the line joining $X = I$ to A, the median AP, is also the angle bisector of $\angle BAC$. We claim that $AP \perp BC$, *i.e.* AP is also an altitude.

Assume, by way of contradiction, that AP is not perpendicular to BC. Let S and T be points on (extensions of) AB and AC, respectively, such that P is on ST and $ST \perp AP$. Since B, P, and C are collinear and BC is not perpendicular to AP, it must be the case that either B is between S and A and T is between C and A, or S is between B and A and C is between T and A. Suppose, for the sake of argument, that it is the latter. (The former case works out in the same way.)



Since $\angle SAP = \angle BAP = \angle CAP = \angle TAP$ by hypothesis, $\angle SPA = \angle TPA$ because both are right angles (as $AP \perp ST$), and $|AP| = |AP|$, the Angle-Side-Angle congruence criterion tells us that $\triangle SAP \cong \triangle TAP$. It follows, in particular, that $|SP| = |TP|$ and $\angle ASP = \angle ATP$.

Besides $|SP| = |TP|$, we have $|BP| = |CP|$ because P is the midpoint of BC, and $\angle BPS = \angle CPT$ because these are opposite angles, so $\triangle BPS \cong \triangle CPS$ by the Side-Angle-Side congruence criterion. It follows that $\angle PBS = \angle PCT$ and $\angle BSP = \angle PTC$.

Thus $\angle BSP = \angle PTC = \angle ATP = \angle ASP$, so, because $\angle BSP + \angle ASP$ makes a straight angle, we have that $\angle BSP$ and $\angle ASP$ are right angles. This is impossible because it would give $\triangle ASP$ two internal angles that are right, and hence a sum of internal angles exceeding a straight angle.

Hence, by contradiction, it must be the case that $AP \perp BC$. Since $\angle BAP = \angle CAP$ (recall that AP is also the angle bisector of $\angle BAC$), $\angle APB = \angle APC$ (because $AP \perp BC$), and $|AP| = |AP|$, the Angle-Side-Angle congruence criterion tells us that $\triangle BAP \cong \triangle CAP$. It follows, in particular, that $|AB| = |AC|$, so the triangle is isosceles. ■

NOTE: I hallucinated a short, sweet, argument when I made up the quiz ... Sorry!

Quiz 10. Thursday, 28 March, 2013. *(10 minutes)*

1. Suppose that the Euler line of $\triangle ABC$ includes the vertex A. Show that the triangle is isosceles. *[5]*

SOLUTION. The line joining vertex A to the orthocentre H is the altitude of the triangle from A, and the line joining A to the centroid is the median from A. By definition, the Euler line includes both the orthocentre H and centroid G of $\triangle ABC$. If it also includes the vertex A, then the Euler line must be both the altitude and the median of the triangle from A. Hence, if P is the point where this line meets BC, then $\angle APB = \angle APC$ are right angles and $|BP| = |CP|$. Since $|AP| = |AP|$ no matter what, it follows by the Side-Angle-Side congruence criterion that $\triangle APB \cong \triangle APC$. This, in turn, implies that $|AB| = |AC|$, so $\triangle ABC$ is isosceles. ■

Quiz 11. Friday, 5 April, 2013. *(10 minutes)*

1. Suppose the nine-point circle of $\triangle ABC$ is tangent to the side BC of the triangle. Show that $\triangle ABC$ is isosceles. *[5]*

SOLUTION. The nine-point circle of $\triangle ABC$ normally passes through two points of BC: the midpoint of the side and the foot of the altitude from A. If the circle is tangent to BC, these two must be the same point, call it P, so AP is both the median from A and the altitude from A. Then $|AP| = |AP|$, $\angle APB = \angle APC =$ a right angle (because $AP \perp BC$), and $|BP| = |CP|$ (because P is the midpoint of BC), so we have $\triangle APB \cong \triangle APC$ by the Side-Angle-Side congruence criterion. Thus $|AB| = |AC|$, so $\triangle ABC$ is isosceles. ■