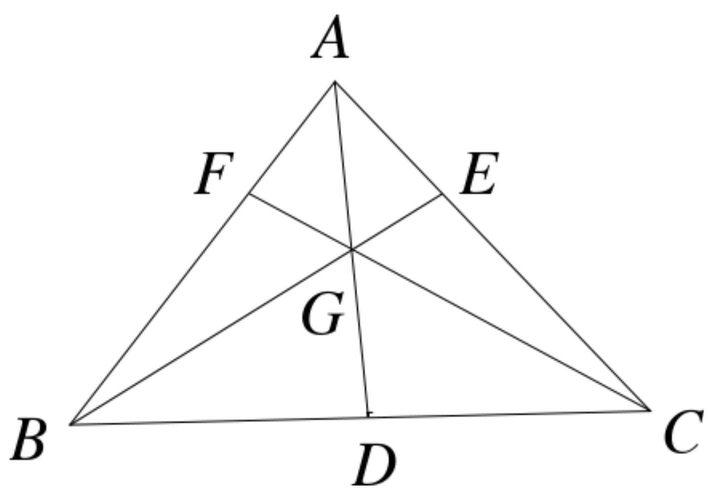
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2012

Solutions to Assignment #6
Ceva's Theorem

The following result appears to have been first obtained by the Arab mathematician Yusuf ibn Ahmad al-Mu'taman ibn Hud, who also served as the ruler of the Emirate of Zaragoza from 1082 to 1085. It was later rediscovered by an Italian Jesuit, Giovanni Ceva (1647-1734), who also rediscovered Menelaus' Theorem.

CEVA'S THEOREM: Suppose D, E, and F are points on the sides BC, AC, and AB, respectively, of $\triangle ABC$. Then AD, BE, and CF all meet in a single point G if and only if $\dfrac{AF}{FB} \cdot \dfrac{BD}{DC} \cdot \dfrac{CE}{EA} = 1$.



1. Prove Ceva's Theorem. *[8 = 2 × 4 each for each direction]*

HINT: ($\Longrightarrow$) You may exploit the fact that the areas of two triangles with the same height are in the same proportion as their bases. Recast the product of ratios as a product of ratios of areas of subtriangles in two different ways, and from there recast is as a third product of ratios of areas of subtriangles.

($\Longleftarrow$) Let G be the intersection of AD and BE and extend CG until it intersects AB at H. Use the $\Longrightarrow$ to help show that $H = F$.

SOLUTION. ($\Longrightarrow$) If we think of $\triangle AGF$ and $\triangle BGF$ as having AF and FB as their bases, it is clear that they have the same height, so $\dfrac{\text{area}\,(\triangle AGF)}{\text{area}\,(\triangle BGF)} = \dfrac{AF}{FB}$. A similar argument shows that $\dfrac{\text{area}\,(\triangle ACF)}{\text{area}\,(\triangle BCF)} = \dfrac{AF}{FB}$ as well, and further similar arguments will also give us $\dfrac{\text{area}\,(\triangle BGD)}{\text{area}\,(\triangle CDG)} = \dfrac{BD}{DC} = \dfrac{\text{area}\,(\triangle ABD)}{\text{area}\,(\triangle CAD)}$ and $\dfrac{\text{area}\,(\triangle CGE)}{\text{area}\,(\triangle AGE)} = \dfrac{CE}{AE} = \dfrac{\text{area}\,(\triangle BCE)}{\text{area}\,(\triangle ABE)}$.

Note that if $\frac{a}{b} = q = \frac{c}{d}$ and $a \neq c$, then $q = \frac{a-c}{b-d}$ as well. It therefore follows from the information obtained above that

$$\frac{AF}{FB} = \frac{\text{area}\,(\triangle ACF) - \text{area}\,(\triangle AGF)}{\text{area}\,(\triangle BCF) - \text{area}\,(\triangle BGF)} = \frac{\text{area}\,(\triangle ACG)}{\text{area}\,(\triangle BCG)}\,,$$
$$\frac{BD}{DC} = \frac{\text{area}\,(\triangle BAD) - \text{area}\,(\triangle BGD)}{\text{area}\,(\triangle CAD) - \text{area}\,(\triangle CGD)} = \frac{\text{area}\,(\triangle ABG)}{\text{area}\,(\triangle ACG)}\,,$$
$$\text{and} \quad \frac{CE}{AE} = \frac{\text{area}\,(\triangle BCE) - \text{area}\,(\triangle CGE)}{\text{area}\,(\triangle ABE) - \text{area}\,(\triangle AGE)} = \frac{\text{area}\,(\triangle BCG)}{\text{area}\,(\triangle ABG)}\,.$$

Hence

$$\begin{aligned}\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} &= \frac{\text{area}\,(\triangle ACG)}{\text{area}\,(\triangle BCG)} \cdot \frac{\text{area}\,(\triangle ABG)}{\text{area}\,(\triangle ACG)} \cdot \frac{\text{area}\,(\triangle BCG)}{\text{area}\,(\triangle ABG)} \\ &= \frac{\text{area}\,(\triangle ACG)}{\text{area}\,(\triangle ACG)} \cdot \frac{\text{area}\,(\triangle ABG)}{\text{area}\,(\triangle ABG)} \cdot \frac{\text{area}\,(\triangle BCG)}{\text{area}\,(\triangle BCG)} \\ &= 1 \cdot 1 \cdot 1 = 1\,,\end{aligned}$$

as desired. ∎

($\Longleftarrow$) Suppose that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$. Following the hint, let G be the intersection of AD and BE and extend CG until it intersects AB at H. By the argument above, since AD, BE, and CH all intersect in G, we must have $\frac{AH}{HB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 = \frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$. It follows that $\frac{AH}{HB} = \frac{AF}{FB}$; since F and H are both points on AB, this is only possible if $F = H$. Thus AD, BE, and $CF = CH$ all intersect in G. ∎

2. Use Ceva's Theorem to verify that the three *medians* of a triangle (*i.e.* the lines joining each vertex to the midpoint of the opposite side) are *concurrent* (*i.e.* meet at a single point).

NOTE: The point where the three medians of a triangle are concurrent is the *centroid* of the triangle. It is one of several possible "centres" of the triangle; we will encounter several others later.

SOLUTION. Suppose D, E, and F are the midpoints of sides BC, CA, and AB, respectively, of $\triangle ABC$. This means that $AF = FB$, $BD = DC$, and $CE = EA$, so

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1 \cdot 1 \cdot 1 = 1\,,$$

and it follows by Ceva's Theorem that the medians AD, BE, and CF are concurrent. ∎