Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Winter 2012

Solution to Assignment #4
Similarity

Euclid's *Elements* doesn't get into similarity until Book VI, but it's a concept that comes in very handy for some things, a few of which we will hopefully get to cover later. We will therefore start developing it now.

Definition. $\triangle ABC$ is *similar* to $\triangle DEF$, often written as $\triangle ABC \sim \triangle DEF$, if corresponding angles are equal (*i.e.* $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$) and the lengths of corresponding sides are all in the same proportion (*i.e.* $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$).

For this assignment you may use all of Euclid's Postulates, plus all the trigonometry you know. (Or can look up!) The trigonometry is likely to be easier to use.

1. Prove the Side-Side-Side Similarity Criterion. *[3]*

If $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$, then $\triangle ABC \sim \triangle DEF$.

Solution. Let $a = \dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$, so $AB = aDE$, $BC = aEF$, and $AC = aDF$. By the Law of Cosines, we have

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - AB \cdot AC \cdot \cos(\angle BAC) \\ \text{and} \quad EF^2 &= DE^2 + DF^2 - DE \cdot DF \cdot \cos(\angle EDF) \,. \end{aligned}$$

It follows that

$$\begin{aligned} \cos(\angle BAC) &= \frac{AB^2 + AC^2 - BC^2}{AB \cdot AC} = \frac{a^2DE^2 + a^2DF^2 - a^2EF^2}{aDE \cdot aDF} \\ &= \frac{DE^2 + DF^2 - EF^2}{DE \cdot DF} = \cos(\angle EDF) \,, \end{aligned}$$

and so, because every internal angle of a triangle must be greater than zero but less than a straight angle [Why?] and $\cos(\theta)$ is $1-1$ for $0 \leq \theta \leq \pi$, we get $\angle BAC = \angle EDF$.

Similar calculations give us that $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$, too. Since we were give that $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$ to start with, it follows that $\triangle ABC \sim \triangle DEF$ by definition. ■

2. Prove the Angle-Angle Similarity Criterion. *[3]*

If $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$, then $\triangle ABC \sim \triangle DEF$.

Solution. Since $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$ and the sum of the interior angles of a triangle must equal a straight angle, we must also have $\angle BAC = \angle EDF$. By the Law of Sines, we have

$$\frac{\sin(\angle ABC)}{AC} = \frac{\sin(\angle BCA)}{AB} = \frac{\sin(\angle CAB)}{BC}$$

$$\text{and} \quad \frac{\sin(\angle DEF)}{DF} = \frac{\sin(\angle EFD)}{DE} = \frac{\sin(\angle FDE)}{EF}\,.$$

It follows that, for one thing,

$$AB = \frac{\sin(\angle BCA)}{\sin(\angle ABC)} AC \quad \text{and} \quad DE = \frac{\sin(\angle EFD)}{\sin(\angle DEF)} DF\,,$$

so, since $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$,

$$\frac{AB}{DE} = \frac{\frac{\sin(\angle BCA)}{\sin(\angle ABC)} AC}{\frac{\sin(\angle EFD)}{\sin(\angle DEF)} DF} = \frac{\frac{\sin(\angle BCA)}{\sin(\angle ABC)} AC}{\frac{\sin(\angle BCA)}{\sin(\angle ABC)} DF} = \frac{AC}{DF}\,.$$

Similar calculations will also show that it also follows that $\frac{AB}{DE} = \frac{BC}{EF}$ and $\frac{AC}{DF} = \frac{BC}{EF}$. Since corresponding angles of $\triangle ABC$ and $\triangle DEF$ are equal and the lengths of corresponding sides are all in the same proportion, $\triangle ABC \sim \triangle DEF$ by definition. ■

~~2~~ **3.** Prove the Side-Angle-Side Similarity Criterion. *[4]*

If $\angle ABC = \angle DEF$ and $\frac{AB}{DE} = \frac{BC}{EF}$, then $\triangle ABC \sim \triangle DEF$.

Solution. Let $a = \frac{AB}{DE} = \frac{BC}{EF}$, so $AB = aDE$ and $BC = aEF$. By the Law of Cosines, we have

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - AB \cdot BC \cdot \cos(\angle ABC) \\ &= a^2 DE^2 + a^2 EF^2 - aDE \cdot aEF \cdot \cos(\angle DEF) \\ &= a^2 \left(DE^2 + EF^2 - DE \cdot EF \cdot \cos(\angle DEF) \right) \\ &= a^2 DF^2\,, \end{aligned}$$

so $AC = aDF$ and $\frac{AB}{DE} = \frac{BC}{EF} = a = \frac{AC}{DF}$. It follows by the Side-Side-Side Similarity Criterion (see question **1**) that $\triangle ABC \sim \triangle DEF$. ■