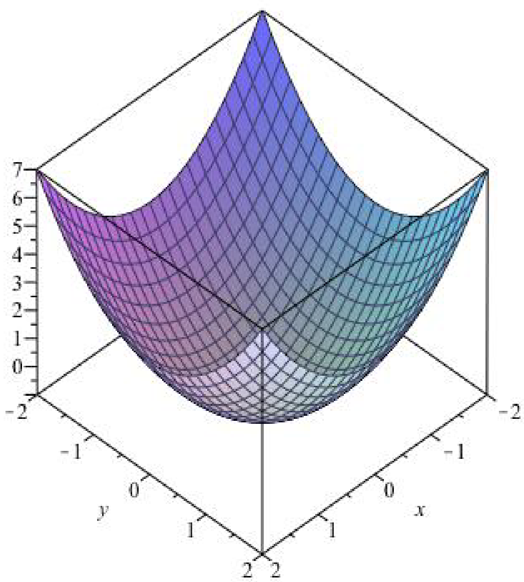
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2012

Solutions to Assignment #1
A geometry on a paraboloid

We will define a geometry $\mathcal{G}$ of points and lines on (the surface of) the paraboloid $z = x^2 + y^2 - 1$, part of which pictured below. (The diagram was generated using `Maple` with the command: `plot3d(x^2+y^-1,x=-2..2,y=-2..2,axes=boxed);`)



The *points* of the geometry $\mathcal{G}$ are the points of the paraboloid, *i.e.* the points (x, y, z) in three-dimensional Cartesian coordinates which satisfy the equation $z = x^2+y^2-1$. The *lines* of the geometry $\mathcal{G}$ are the curves on the paraboloid obtained by intersection a plane through the origin in three-dimensional space with the paraboloid. (Note that every plane through the origin does indeed intersect the paraboloid.) Your task will be to determine some of the basic properties of $\mathcal{G}$.

1. Suppose P and Q are two distinct points of $\mathcal{G}$. Show that there is a unique line ℓ of $\mathcal{G}$ which passes through both P and Q. *[3]*

SOLUTION. This is true most of the time, but not always:

First, if the two points P and Q of $\mathcal{G}$, considered as points of $\mathbb{R}^3$, are not in a straight line with the origin, then there is only one plane passing through all three. The intersection of this plane with the paraboloid is thus the only line of $\mathcal{G}$ joining P and Q.

However, if the two points P and Q of $\mathcal{G}$ are in a straight line with the origin, then there are infinitely many planes passing through all three (which all intersect in that straight line). The intersection of each of these infinitely many planes with the paraboloid gives a (different!) line of $\mathcal{G}$ joining P and Q. ■

2. Give (different! :-) examples to show that it is possible for distinct lines ℓ and m of $\mathcal{G}$ to intersect in 0, 1, or 2 points of $\mathcal{G}$. *[3]*

SOLUTION. *i. (An example with* 2 *points.)* Let ℓ be the line of $\mathcal{G}$ obtained by intersecting the plane $x = 0$ of $\mathbb{R}^3$ with the paraboloid $z = x^2 + y^2 - 1$, and let m be the line of $\mathcal{G}$ obtained by intersecting the plane $z = 0$ of $\mathbb{R}^3$ with the paraboloid. Then ℓ and m meet at two points on the paraboloid: $(0, 1, 0)$ and $(0, -1, 0)$. (You can check for yourselves that these are the only points on all three of $x = 0$, $z = 0$, and $z = x^2 + y^2 - 1$.) □

ii. (An example with just 1 *point.)* Let ℓ be the line of $\mathcal{G}$ obtained by intersecting the plane $x = 0$ of $\mathbb{R}^3$ with the paraboloid $z = x^2 + y^2 - 1$, and let m be the line of $\mathcal{G}$ obtained by intersecting the plane $y = 0$ of $\mathbb{R}^3$ with the paraboloid. Then ℓ and m meet at the tip of the paraboloid the point, *i.e.* at $(0, 0, -1)$, and at no other point. (You can check for yourselves that this is the only point on all three of $x = 0$, $y = 0$, and $z = x^2 + y^2 - 1$.) □

ii. (There are no examples with 0 *points.)* Suppose we have distinct lines ℓ and m of $\mathcal{G}$, which are obtained by intersecting the planes **L** and **M** through the origin, respectively, with the paraboloid. Since these two planes must be different and both pass through the origin, they intersect in a straight line of $\mathbb{R}^3$ which passes through the origin. However, any straight line through the origin must intersect the paraboloid at least once. (I'll leave this to you to check!) A point where the straight line meets the paraboloid is a point where the planes **L** and **M** both intersect the paraboloid, and hence is a point that the lines ℓ and m of $\mathcal{G}$ have in common. ■

3. Explain why distinct lines ℓ and m of $\mathcal{G}$ cannot intersect in 3 points of $\mathcal{G}$. *[1]*

SOLUTION. It is enough to show that if lines ℓ and m of $\mathcal{G}$ intersect in 3 different points of $\mathcal{G}$, say P, Q, and R, then $\ell = m$; *i.e.* that the two lines of $\mathcal{G}$ are not distinct. Note that since any straight line through the origin in $\mathbb{R}^3$ intersects the paraboloid at most twice, some subset of three of P, Q, R, and the origin (considered as points of $\mathbb{R}^3$) are not all on a straight line.

Now ℓ is obtained by intersecting the paraboloid with a plane, call it **L**, through the origin; then P, Q, and R, considered as points of $\mathbb{R}^3$, must be on the plane **L** as well. Similarly, m is obtained by intersecting the paraboloid with a plane, call it **M**, through the origin, and P, Q, and R, considered as points of $\mathbb{R}^3$, are on the plane **M** as well. Since some subset of three of P, Q, R, and the origin are not all on a straight line, it follows that there is only one plane that passing through all of them. Hence $\mathbf{L} = \mathbf{M}$, from which it follows that $\ell = m$. ■

4. Determine whether or not *Playfair's Axiom*,

> If P is a point not on the line ℓ, then there is a unique line m through P such that l and m do not intersect in any point.

is true in $\mathcal{G}$. *[3]*

SOLUTION. As was noted in the solution to **2**, it is not actually possible to have two lines of $\mathcal{G}$ that do not intersect. It follows that there are no parallel lines in $\mathcal{G}$, so Playfair's Axiom cannot hold. ■