Mathematics 2260H – Geometry I: Euclidean geometry

TRENT UNIVERSITY, Winter 2012

Quiz Solutions

Quiz #1. Tuesday, 17 January, 2012. *[10 minutes]*

1. Given a line segment AB, use (some of) Postulates I–V, A, and S to show there exists a line segment that is exactly three times the length of AB. *[5]*

SOLUTION. Here is a step-by-step construction of such a line segment:

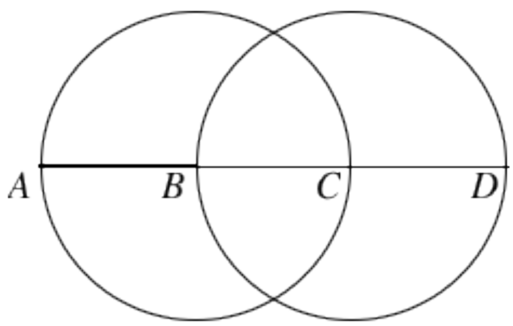
i. Draw a circle with centre B and radius AB (Postulate III).

ii. Extend AB past B until it meets the circle drawn in step i (Postulates II and S). Call this point of intersection C.

iii. Draw a circle with centre C and radius BC (Postulate III).

iv. Extend AC past C until it meets the circle drawn in step iii (Postulates II and S). Call this point of intersection D.

Here's a diagram of the construction:

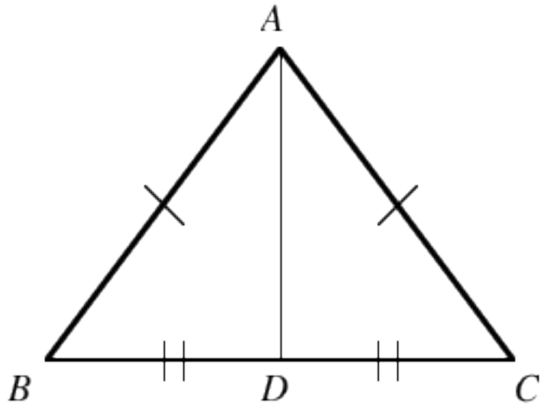


AD is the segment we want. Note that $AB = BC$ because both are radii of the circle drawn in step i and that $BC = CD$ because both are radii of the circle drawn in step iii. It follows (by one of the "Common Notions") that $AB = CD$ as well. Thus $AD = AB + BC + CD$ is three times the length of AB. ■

Quiz #2. Tuesday, 24 January, 2012. *[10 minutes]*

1. Suppose D is the midpoint of side BC (*i.e.* $BD = CD$) of $\triangle ABC$ and $AB = AC$. Show that $\angle ADB$ is a right angle. *[5]*

SOLUTION. Here's a picture of the given situation:



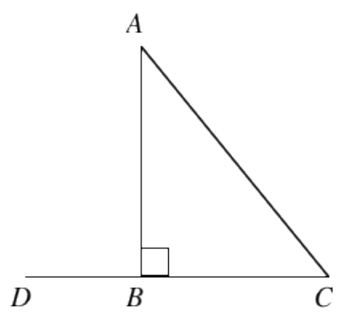
Recall that, by definition, a right angle occurs when a line falls on another line and the two angles it makes on the same side of the line it falls across are equal. Since $\angle ADB$ and $\angle ADC$ occur on the same side when AD falls across BC, it is enough to show that $\angle ADC = \angle ADB$; both must then be right angles.

By hypothesis, we have $AB = AC$ and $BD = CD$. Since we also have $AD = AD$ – things are equal to themselves! – it follows by the Side-Side-Side congruence criterion that $\triangle ABD \cong \triangle ACD$. Hence corresponding angles in the two triangles must also be equal, so $\angle ADB = \angle ADC$, as required. Thus $\angle ADB$ is a right angle. ■

Quiz #3. Tuesday, 31 January, 2012. *[10 minutes]*

1. Suppose $\angle ABC$ of $\triangle ABC$ is a right angle. Show that $\angle ACB$ is *not* a right angle. (Without using Postulate V or any equivalent of it ...) *[5]*

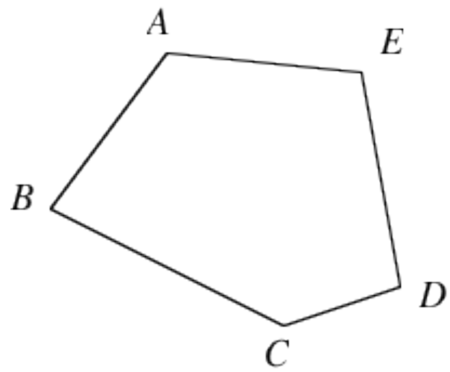
SOLUTION. Extend CB past B to some point D.



Since $\angle ABC$ is a right angle and $\angle ABC + \angle ABD$ is a straight angle, it follows from the definition of "right angle" that $\angle ABD$ is a also a right angle. By Proposition I-16, the exterior angle $\angle ABD$ (which is a right angle) is greater than the opposite interior angle $\angle ACB$. It follows that $\angle ABC \neq \angle ABD$, so, since all right angles are equal by Postulate IV, $\angle ABC$ is not a right angle. ■

Quiz #4. Tuesday, 7 February, 2012. *[10 minutes]*

1. Suppose $ABCDE$ is a (convex) pentagon, as in the diagram below.



Show that the sum of the interior angles of $ABCDE$ is equal to six right angles. *[5]*

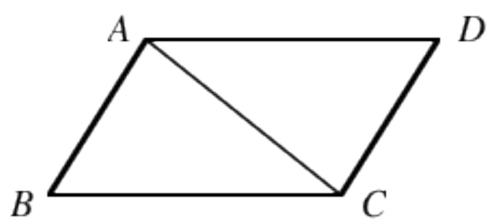
Solution. Connect A to C and to D. The sum of the interior angles of each of $\triangle ABC$, $\triangle ACD$, and $\triangle ADE$ is two right angles. We now compute the sum of the interior angles of the pentagon:

$$\begin{aligned}
&\angle ABC + \angle BCD + \angle CDE + \angle DEA + \angle EAB \\
&= \angle ABC + (\angle BCA + \angle ACD) + (\angle CDA + \angle ADE) \\
&\qquad + \angle DEA + (\angle EAD + \angle DAC + \angle CAB) \\
&= (\angle ABC + \angle BCA + \angle CAB) + (\angle ACD + \angle CDA + \angle DAC) \\
&\qquad + (\angle ADE + \angle DEA + \angle EAD) \\
&= 2 \text{ right angles} + 2 \text{ right angles} + 2 \text{ right angles} \\
&= 6 \text{ right angles} \qquad \blacksquare
\end{aligned}$$

Quiz #5. Tuesday, 7 February, 2012. *[10 minutes]*

1. Suppose $ABCD$ is a convex quadrilateral such that $\triangle ABC \cong \triangle CDA$. Show that $ABCD$ is a parallelogram.

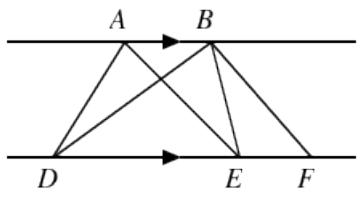
Solution. Here's a sketch of the situation:



Since $\triangle ABC \cong \triangle CDA$, we must have that $BC = DA$ and $\angle BCA = \angle DAC$. By the Z-theorem, it follows from the latter that $BC \parallel AD$. Similarly, we must also have $AB = CD$ and $\angle BAC = \angle DCA$; with the Z-theorem then implying that $AB \parallel CD$. Thus opposite sides of the quadrilateral $ABCD$ are parallel and equal in length, so $ABCD$ is a parallelogram. $\blacksquare$

Quiz #6. Tuesday, 28 February, 2012. *[10 minutes]*

1. Suppose $AB \parallel DF$ and E is on DF between D and F.



Show that $\text{area}(\triangle ADE) + \text{area}(\triangle BEF) = \text{area}(\triangle BDF)$. *[5]*

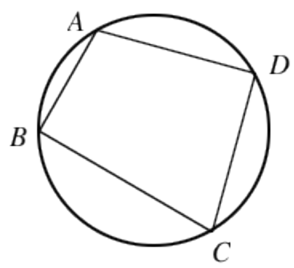
Solution. Note that $\text{area}(\triangle ADE) = \text{area}(\triangle BDE)$ because these triangles are on the same base (DE) and in the same parallels (Euclid's Proposition I-37). Then, since $\triangle BDF$ can be partitioned or dissected into $\text{area}(\triangle BDE)$ and $\text{area}(\triangle BEF)$,

$$\text{area}(\triangle BDF) = \text{area}(\triangle BDE) + \text{area}(\triangle BEF) = \text{area}(\triangle ADE) + \text{area}(\triangle BEF)\,,$$

as desired. ■

Quiz #7. Tuesday, 6 March, 2012. *[10 minutes]*

1. Suppose $ABCD$ is a convex quadrilateral inscribed in a circle.



Show that $\angle ABC + \angle CDA = 2$ right angles. *[5]*

Solution. Let O be the centre of the circle. Then $\angle ABC = \frac{1}{2}\angle AOC$ and $\angle CDA = \frac{1}{2}\angle COA$. It follows that

$$\begin{aligned}\angle ABC + \angle CDA &= \frac{1}{2}\angle AOC + \frac{1}{2}\angle COA = \frac{1}{2} \cdot 2 \cdot \text{straight angles}\\ &= 1 \text{ straight angle} = 2 \text{ right angles}\,,\end{aligned}$$

as desired. ■

Quiz #8. Tuesday, 13 March, 2012. *[10 minutes]*

1. Suppose the centroid and orthocentre of $\triangle ABC$ are the same point. Show that $\triangle ABC$ is equilateral.

NOTE: Recall that the centroid of a triangle is the point where the three medians – the lines joining each vertex to the midpoint of the opposite site – meet, and that the orthocentre is the point where the three altitudes of the triangle meet.

SOLUTION. Let O be the centroid/orthocentre of $\triangle ABC$, and let D be the point where AO meets BC. Since O is the centroid of $\triangle ABC$, AD must be the median from A, and so D is the midpoint of BC, *i.e.* $BD = DC$. Also, since O is the orthocentre of $\triangle ABC$, AD must be the altitude from A, and so $\angle ADB = \angle ADC$ are right angles. Since $AD = AD$, it follows by the side-angle-side congruence criterion that $\triangle ADB \cong \triangle ADC$, so $AB = AC$. Similar arguments starting from BO and CO show that $BA = BC$ and $CA = CB$, so $AB = BC = AC$, *i.e.* $\triangle ABC$ is equilateral. ■

Quiz #9. Tuesday, 20 March, 2012. *[10 minutes]*

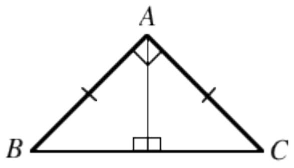
1. Suppose the incentre of $\triangle ABC$ is on the altitude from A. Show that $\triangle ABC$ is isosceles.

NOTE: Recall that the incentre is the point where the three angle-bisectors of the triangle meet.

SOLUTION. Suppose P is the point where the altitude from A meets the side BC. Since AP is an altitude, $\angle APB = \angle APC =$ a right angle. AP also passes through the incentre, so it must be the line bisecting $\angle BAC$, *i.e.* $\angle BAP = \angle CAP$. Since we obviously have $AP = AP$, $\triangle APB \cong \triangle APC$. It follows that $AB = AC$, so $\triangle ABC$ is isosceles. ■

Quiz #10. Tuesday, 27 March, 2012. *[10 minutes]*

1. Suppose $\triangle ABC$ is an isosceles triangle with $\angle BAC$ a right angle.



Show that the Euler line of $\triangle ABC$ is the altitude from A. *[5]*

SOLUTION. The orthocentre of $\triangle ABC$ is the intersection point of all three altitudes. Since $\angle BAC$ is a right angle, AB and AC are altitudes of $\triangle ABC$, so their point of intersection, A, is the orthocentre of $\triangle ABC$.

Since $\triangle ABC$ is isosceles with $\angle BAC$ a right angle, the two short sides must be equal, because neither can be equal to the hypotenuse. It follows that $AB = AC$ and that $\angle ABC = \angle ACB$. If D is the point where the altitude from A meets BC, then we also have that $\angle ADB = \angle ADC$, since both are right angles. It follows by the angle-angle-side congruence crierion that $\triangle ABD \cong \triangle ACD$, from which we can conclude that $BD = CD$, *i.e.* D is the midpoint of BC.

It follows from the above that AD is the median from A and the perpendicular bisector of BC, as well as the altitude from A, so the circumcentre and the centroid of $\triangle ABC$ must be on AD as well – we already know the orthocentre is – *i.e.* AD is the Euler line of $\triangle ABC$.

It's probably worth noticing that A can be neither the circumcentre nor the centroid: since A is not the midpoint of AB or of AC, the medians from B and C and the perpendicular bisectors of AB and AC cannot pass through A. ■