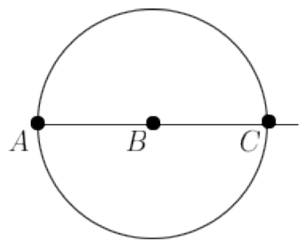
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2011

Quiz Solutions

Quiz #1. ~~Wednesday, 19~~ Thursday, 20 January, 2011 *[10 minutes]*

1. Given a line segment AB, construct a point C so that B is on AC and the length of AC is twice the length of AB. *[5]*

SOLUTION. Draw a circle with radius AB and centre B [Postulate III]. Extend AB is straight line past B until its end is outside the circle [Postulate II]. The line must then meet the circle [Postulate S]; call this point C.

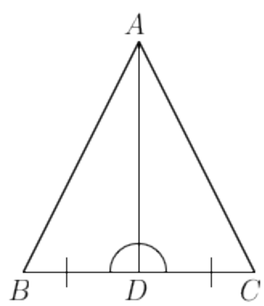


By the construction, B is on the line AC, and since BC is also a radius of the circle, we have $AB = BC$. It follows that $AC = AB + BC = 2AB$, as desired. ■

Quiz #2. Wednesday, 26 January, 2011 *[10 minutes]*

1. Suppose D is the midpoint of the side BC of $\triangle ABC$ and $\angle ADB = \angle ADC$. Show that $AB = AC$.

SOLUTION. Since D is the midpoint of BC, $DB = DC$.

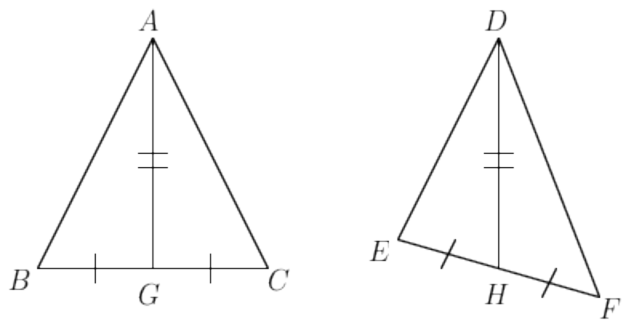


Since we are given that $\angle ADB = \angle ADC$, and we also have $AD = AD$ (Why?), it follows by the side-angle-side congruence criterion [Proposition I-4] that $\triangle ADB \cong \triangle ADC$. In particular, this means that the corresponding sides AB and AC are equal, as desired. ■

Quiz #3. ~~Wednesday, 2~~ ~~Thursday, 3~~ Monday, 7 February, 2011 *[10 minutes]*

1. Suppose that in $\triangle ABC$ and $\triangle DEF$, G and H are the midpoints of BC and EF, respectively, and that $AG = DH$ and $BG = EH$. Use an example to show that the given triangles do not have to be congruent. *[5]*

Solution. Here is pretty simple way to make an example. Start with a $\triangle ABC$ with midpoint G of side BC. Make a congruent copy $\triangle DEF$ of $\triangle ABC$ with midpoint H of side EF, and then pivot EF about H so that E is moved a little closer to and F a little farther from D:

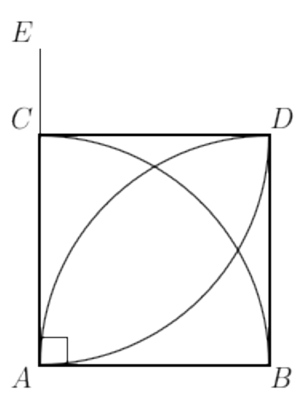


This does not change the conditions that G and H are the midpoints of BC and EF, respectively, and that $AG = DH$ and $BG = EH$, but it does make $DE < AB$ and $DF > AC$, so $\triangle ABC \ncong \triangle DEF$. ■

Quiz #4. Wednesday, 9 February, 2011 *[10 minutes]*

1. Given a line segment AB, construct a quadrilateral with four equal sides, one of which is AB, and a right angle at A. *[5]*

Solution. Construct a line segment AE perpendicular to AB. Draw a circle of radius AB centred at A, and let C be the point where this circle intersects AE (extended, if necessary) on the same side as E. (Note that it might, accidentally, happen that $C = E$.) Now draw a circle of radius BA centred at B and a circle of radius CA centred at C, and let D be the point other than A where these two circles intersect.

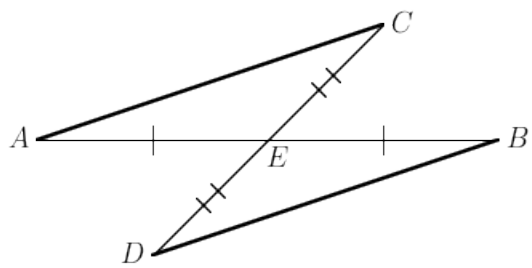


Then $AB = BD = DC = CA$ and $\angle BAC$ is a right angle, as required. ■

Quiz #5. ~~Wednesday, 16~~ Thursday, 17 February, 2011 *[10 minutes]*

1. Suppose line segments AB and CD each bisect the other at their intersection point E. Show that AC is parallel to BD. *[5]*

SOLUTION. Note that $\angle AEC = \angle BED$ since they are opposite angles. Since AB and CD bisect each other at E, we also have $AE = EB$ and $CE = ED$. Hence $\triangle AEC \cong \triangle BED$ by the side-angle-side congruence criterion.

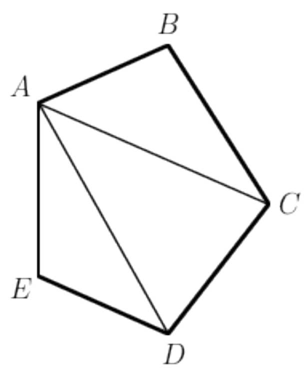


It follows that $\angle ACD = \angle ACE = \angle BDE = \angle BDC$, so $AC \parallel BD$ by (half of) the Z-theorem (*i.e.* by I-28). ■

Quiz #6. Some day or other. *[10 minutes]*

1. Assuming that the sum of the interior angles of a triangle is equal to two right angles, show that the sum of the interior angles of a (convex!) pentagon is equal to six right angles. *[5]*

SOLUTION. Suppose $ABCDE$ is a convex pentagon. Connect A to the other vertices of the pentagon that it is not already connected to, as in the diagram below. Note that since the pentagon is convex, each of these ne line segments is contained within the pentagon.



This divides up the pentagon into three triangles. Since the sum of the interior angles of a triangle is equal to two right angles, *i.e.* π radians, we have that:

$$\begin{aligned}\pi &= \angle ABC + \angle BCA + \angle CAB \\ &= \angle ACD + \angle CDA + \angle DAC \\ &= \angle ADE + \angle DEA + \angle EAD\end{aligned}$$

It follows that

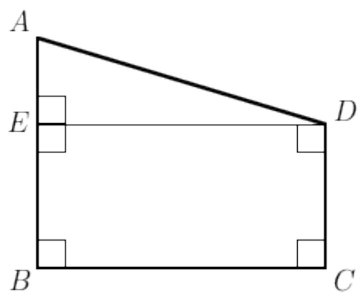
$$
\begin{aligned}
&\angle ABC + \angle BCD + \angle CDE + \angle DEA + \angle EAB \\
&= \angle ABC + (\angle BCA + \angle ACD) + (\angle CDA + \angle ADE) + \angle DEA \\
&\qquad + (\angle EAD + \angle DAC + \angle CAB) \\
&= (\angle ABC + \angle BCA + \angle CAB) + (\angle ACD + \angle CDA + \angle DAC) \\
&\qquad + (\angle ADE + \angle DEA + \angle EAD) \\
&= 3\pi\,,
\end{aligned}
$$

which is equal to six right angles, as desired. ■

Quiz #7. Wednesday, 9 March, 2011. *[10 minutes]*

1. Suppose $ABCD$ is a quadrilateral such that $\angle ABC$ and $\angle BCD$ are right angles. Show that the area of $ABCD$ is equal to $\frac{1}{2}(AB + CD) \cdot BC$. *[5]*

SOLUTION. Assume, without loss of generality, that $AB \geq CD$. Draw a line parallel to BC through D, which meets AB at E.



Since $DE \parallel BC$, $\angle CDA$, $\angle DEB$, and $\angle DEA$ are all right angles, and it is easy to see that $DE = BC$ and $BE = CD$. Thus $EBCD$ is a rectangle, and hence has area $CD \cdot BC$, and $\triangle AED$ is right triangle, and hence has area $\frac{1}{2} AE \cdot ED$. Then

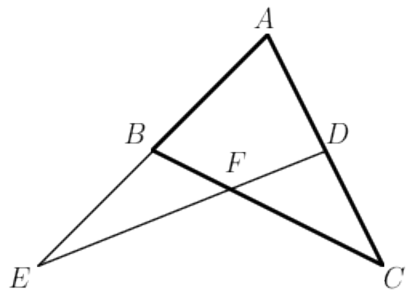
$$
\begin{aligned}
\text{area } ABCD &= \text{area } EBCD + \text{area } \triangle AED \\
&= CD \cdot BC + \frac{1}{2} AE \cdot ED = CD \cdot BC + \frac{1}{2} AE \cdot BC \\
&= \frac{1}{2}(CD + EB) \cdot BC + \frac{1}{2} AE \cdot BC \\
&= \frac{1}{2}(CD + EB + AE) \cdot BC = \frac{1}{2}(CD + AB) \cdot BC\,,
\end{aligned}
$$

as desired. ■

Quiz #8. Wednesday, 16 March, 2011. *[10 minutes]*

1. Suppose D, E, and F are collinear points on the side AC, AB, and BC, respectively, of $\triangle ABC$, and $AD = DC$ and $AB = BE$. Compute $\dfrac{CF}{FB}$. *[5]*

SOLUTION. By Menelaus' Theorem, D, E, F collinear implies $\dfrac{AE}{EB} \cdot \dfrac{BF}{FC} \cdot \dfrac{CD}{DA} = -1$.



From $AB = BE$ it follows that $AE = AB + BE = 2BE$, and hence that $\dfrac{AE}{EB} = \dfrac{2}{-1} = -2$. Similarly, it follows from $AD = DC$ that $\dfrac{CD}{DA} = \dfrac{1}{1} = 1$. Thus

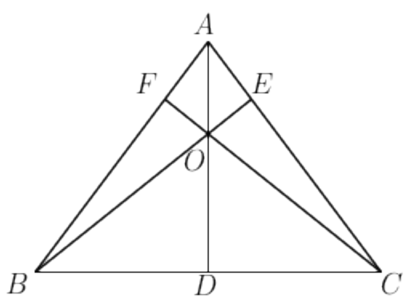
$$-1 = \frac{AE}{EB} \cdot \frac{BF}{FB} \cdot \frac{CD}{DA} = (-2) \cdot \frac{BF}{FC} \cdot 1\,,$$

so $\dfrac{BF}{FC} = \dfrac{1}{2}$. Hence $\dfrac{CF}{FB} = 1/\left(\dfrac{-BF}{-FC}\right) = 1/\left(\dfrac{BF}{FC}\right) = 1/\left(\dfrac{1}{2}\right) = 2$. ■

Quiz #9. Wednesday, 23 March, 2011. *[12 minutes]*

1. Suppose O is the orthocentre of $\triangle ABC$, *i.e.* the point where the three altitudes meet. Show that A is the orthocentre of $\triangle OBC$. *[5]*

SOLUTION. Let D, E, and F be the points where the altitudes from A, B, and C, respectively, meet the opposite sides of the triangle.



This means, by the definition of altitude, that $AD \perp BC$, $BE \perp AC$, and $CF \perp AB$. Since the altitudes all pass through the orthocentre, it follows that $OD \perp BC$, $BO \perp CA$, and $CO \perp BA$, *i.e.* OD, CA, and BA are (extensions of) the altitudes of $\triangle OBC$. Since all three of these altitudes pass through A, A is the orthocentre of $\triangle OBC$. ■

Quiz #10. Wednesday, 30 March, 2011. *[10 minutes]*

1. Suppose that the orthocentre (where the three altitudes meet) and the incentre (where the three angle bisectors meet) of $\triangle ABC$ are the same point. Show that $\triangle ABC$ is equilateral. *[5]*

SOLUTION. Suppose O is the point which is both the orthocentre and the incentre of $\triangle ABC$, and D is the point where AO meets BC. Since O is the orthocentre, OD is an altitude of the triangle and so $\angle ODB$ and $\angle ODC$ are both right angles. Since O is the incentre as well, AD is also an angle bisector, so $\angle BAD = \angle CAD$. Because $OD = OD$, it follows that $\triangle ADB \cong \triangle ADC$ by angle-side-angle congruence. This, in turn, implies that $AB = AC$.

We can repeat the argument above for the altitudes/angle-bisectors from B and C to get $AB = BC$ and $AC = BC$, so $AB = AC = BC$, *i.e.* the triangle is equilateral. ■

Quiz #11. Wednesday, 36 April, 2011. *[10 minutes]*

1. Supose D is the circumcentre of $\triangle ABC$ and D is on the same side of BC as A. Show that $\angle BDC = 2\angle BAC$. *[5]*

SOLUTION. A slightly different way of describing this set-up is that D is the centre of a circle and A is a point on the circle on the same side of the chord BC of the circle as D is. By our main result on angles inscribed in circles, it follows that $\angle BAC = \frac{1}{2}\angle BDC$, so $\angle BDC = 2\angle BAC$, as desired. ■