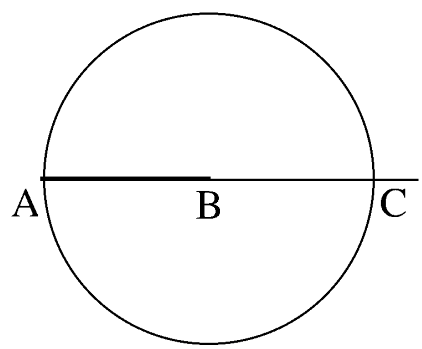
Mathematics 226H – Geometry I: Euclidean geometry
Trent University, Winter 2008

Solutions to Quizzes

Quiz #1. Friday, 18 January, 2008. *[10 minutes]*

1. Given a line segment AB, show, using Euclid's system, that there is a point C so that B is on AC and $|BC| = |AB|$. *[5]*

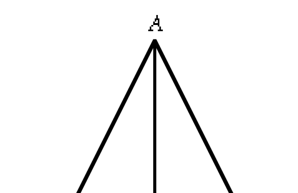
Solution. Suppose we are given a line segment AB. Draw a circle with centre B and radius AB [Postulate 3]. Extend AB past B in a straight line [Postulate 2] until the straight line intersects the circle (on the other side of B from A) [that it should do so is implicit in Definitions 15 and 17]. Let C be this point of intersection.



B, the centre of the circle, is on AC by the construction of AC, and $|BC| = |AB|$ since AB and BC are both radii of the circle [Definition 15]. ■

Quiz #2. Friday, 25 January, 2008. *[10 minutes]*

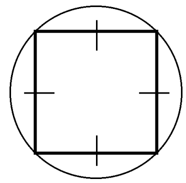
1. Suppose that the median from A in $\triangle ABC$ is also an altitude. Show that $\triangle ABC$ is isosceles. *[5]*



Solution. Let D denote the point where the median from A meets BC. Since AD is a median, $|BD| = |CD|$, and since it is also an altitude, $\angle ADB = \angle ADC = 90°$. As we also have $|AD| = |AD|$ – any line segment is just as long as itself – it follows by the SAS congruence criterion that $\triangle ADB \cong \triangle ADC$. Hence $|AB| = |AC|$ and $\angle ABC = \angle ABD = \angle ACD = \angle ACB$, so $\triangle ABC$ is isosceles. ■

Quiz #3. Friday, 1 February, 2008. *[10 minutes]*

1. Show that a rhombus inscribed in a circle must be a square. *[5]*



Solution. Suppose $ABCD$ is a rhombus inscribed in a circle and O is the centre of the circle. Since OA, OB, OC, and OD are all radii of the circle, we have $|OA| = |OB| = |OC| = |OD|$, and since $ABCD$ is rhombus, we also have $|AB| = |BC| = |CD| = |DA|$. By the SSS congruence criterion it follows that $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong \triangle ODA$. Note also that each of these triangles is isosceles. It follows that $\angle ABC = 2\angle ABO = \angle BCD = \angle CDA = \angle DAB$. Since the angles of a quadrilateral must add up to $360°$, it follows that $\angle ABC = \angle BCD = \angle CDA = \angle DAB = \frac{1}{4}360° = 90°$. Thus the rhombus $ABCD$ is also a rectangle, but a rectangle whose sides are all equal is a square. ■

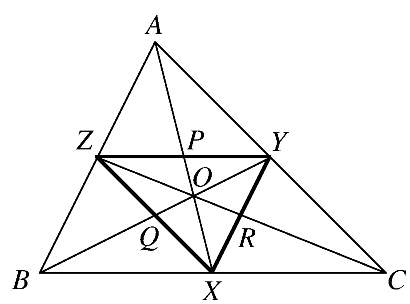
Quiz #4. Friday, 8 February, 2008. *[10 minutes]*

1. Suppose $\triangle ABC$ and $\triangle PQR$ have $\angle A = \angle P = 90°$ and $\frac{|AB|}{|PQ|} = \frac{|BC|}{|QR|}$. Show that $\angle B = \angle Q$. *[5]*

Solution. Let $\alpha = \frac{|AB|}{|PQ|} = \frac{|BC|}{|QR|}$, so $|AB| = \alpha|PQ|$ and $|BC| = \alpha|QR|$. Since $\angle A = \angle P = 90°$, the two triangles are right triangles. By the Pythagorean Theorem, it follows that $|BC|^2 = |AB|^2 + |AC|^2$ and $|QR|^2 = |PQ|^2 + |PR|^2$, so $|AC|^2 = |BC|^2 - |AB|^2 = \alpha^2|QR|^2 - \alpha^2|PQ|^2 = \alpha^2|PR|^2$. Hence $|AC| = \alpha|PR|$, so $\frac{|AC|}{|PR|} = \alpha = \frac{|AB|}{|PQ|} = \frac{|BC|}{|QR|}$. By the side-side-side criterion for similarity, it follows that $\triangle ABC \sim \triangle PQR$, and hence $\angle B = \angle Q$. ■

Quiz #5. Friday, 15 February, 2008. *[10 minutes]*

1. The medians AX, BY, and CZ meet in the centroid O of $\triangle ABC$. Show that O is also the centroid of $\triangle XYZ$. *[5]*



Solution. Let $P = AX \cap YZ$, $Q = BY \cap XZ$, and $R = CZ \cap XY$, as in the diagram above. We will first show that these three points are the midpoints of the sides of $\triangle XYZ$.

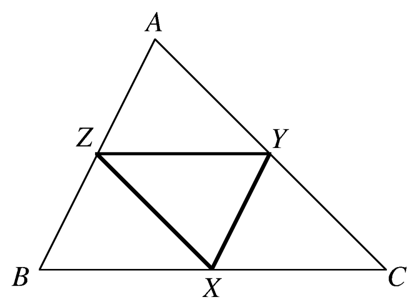
Since Y and Z are the midpoints of AC and AB, respectively, $YZ \parallel BC$ and $|YZ| = \frac{1}{2}|BC|$. $\angle PAY = \angle XAB$ since these are the same angle, and it follows from $YZ \parallel BC$

that $\angle ABX = \angle AYP$. Thus $\triangle APY \sim \triangle AXB$ by the angle-angle similarity criterion, and since $|AY| = \frac{1}{2}|AB|$ (Y being the midpoint of AC) and $|XB|\frac{1}{2}|AB|$ (X being the midpoint of BC, it follows that $|PY| = \frac{1}{2}|XB| = \frac{1}{2} \cdot \frac{1}{2}|AB| = \frac{1}{2}|ZY|$. Thus P is the midpoint of ZY. Similar arguments show that Q is the midpoint of XZ and that R is the midpoint of XY.

It follows from the above that XP, YQ, and ZR are the medians of $\triangle XYZ$, and so their common point of intersection, the centroid O of $\triangle ABC$, is also the centroid of $\triangle XYZ$. ■

Quiz #6. Friday, 7 March, 2008. *[10 minutes]*

1. Suppose X, Y, and Z are the midpoints of sides BC, AC, and AB, respectively, of $\triangle ABC$. Show that the circumcentre of $\triangle ABC$ is also the orthocentre of $\triangle XYZ$. *[5]*



Solution. The altitude of $\triangle XYZ$ from X is, by definition, perpendicular to YZ. Since Y and Z are the midpoints of AC and AB, respectively, $YZ \parallel BC$. Hence the altitude of $\triangle XYZ$ from X is perpendicular to BC and passes through the midpoint of BC, so it is also the perpendicular bisector of BC.

Similarly, the altitudes from Y and Z of $\triangle ABC$ are also the pependicular bisectors of AC and AB respectively. Considered as altitudes of $\triangle XYZ$, the three lines are concurrent in the orthocentre of $\triangle XYZ$, and considered as perpendicular bisectors of the sides of $\triangle ABC$, the same three lines are concurrent in the circumcentre of $\triangle ABC$. Since three lines can be concurrent in at most one point, the orthocentre of $\triangle XYZ$ must also be the circumcentre of $\triangle ABC$. ■

Quiz #7. Friday, 14 March, 2008. *[10 minutes]*

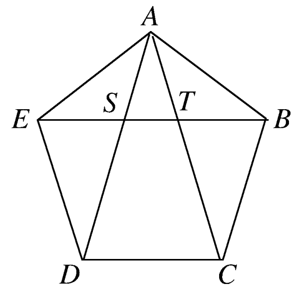
1. Suppose $\triangle ABC$ has $\angle C = 90°$ and sides $a = 3$, $b = 4$, and $c = 5$. Find the inradius r of $\triangle ABC$. *[5]*

Hint: Depending on how you proceed, you may find the trigonometric identity $\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$ to be useful.

Solution. Recall that for any triangle, $K_{ABC} = rs$, where r is the inradius and $s = \frac{a+b+c}{2}$ is the semiperimeter of $\triangle ABC$. In this case, $K_{ABC} = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}4 \cdot 3 = 6$ and $s = \frac{3+4+5}{2} = frac122 = 6$, so $r = K_{ABC}/s = 6/6 = 1$. ■

Quiz #8. Thursday, 20 March, 2008. *[10 minutes]*

1. Suppose $ABCDE$ is a regular pentagon, S is the intersection of AD and BE, and T is the intersection of AC and BD. Compute $\mathbf{cr}(E, S, T, B)$. *[5]*



Hint: The following values of $\sin(\theta)$ may be of use:

θ	$0°$	$36°$	$72°$	$108°$
$\sin(\theta)$	0	0.59	0.95	0.95

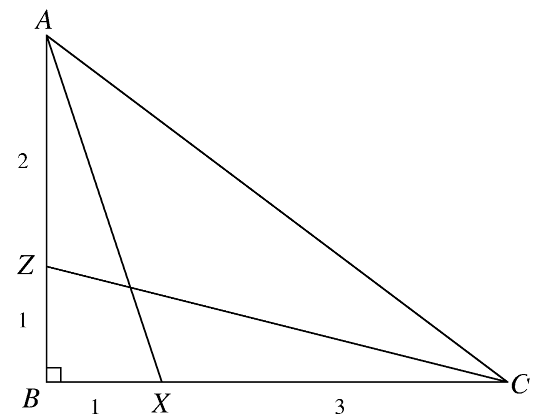
Solution. Since $ABCDE$ is a regular pentagon, its five vertices are cocircular and divide up the circle in question into five equal arcs. Note that the collinear points E, S, T, and B are in perspective from A with the points E, D, C, and B, which are cocircular with A. Hence,

$$\begin{aligned}
\mathbf{cr}(E,S,T,B) &= \mathbf{cr}(E,D,C,B) \\
&= \frac{\sin\left(\frac{1}{2}\mathrm{arc}(EC)\right)\sin\left(\frac{1}{2}\mathrm{arc}(DB)\right)}{\sin\left(\frac{1}{2}\mathrm{arc}(EB)\right)\sin\left(\frac{1}{2}\mathrm{arc}(DC)\right)} \\
&= \frac{\sin\left(\frac{1}{2}\cdot\frac{2}{5}\cdot 360°\right)\sin\left(\frac{1}{2}\cdot\frac{2}{5}\cdot 360°\right)}{\sin\left(\frac{1}{2}\cdot\frac{3}{5}\cdot 360°\right)\sin\left(\frac{1}{2}\cdot\frac{1}{5}\cdot 360°\right)} \\
&= \frac{\sin(72°)\sin(72°)}{\sin(108°)\sin(36°)} \\
&= \frac{0.95\cdot 0.95}{0.95\cdot 0.59} = \frac{0.95}{0.59} = 1.61.
\end{aligned}$$

(Approximately, of course!) ∎

Quiz #9. Friday, 28 March, 2008. *[10 minutes]*

1. Suppose $\triangle ABC$ is a right triangle with $\angle B = 90°$, $a = 4$, $b = 5$, and $c = 3$. Z isa point on side AB such that $|AZ| = 2$, and X is a point on side BC such that $|BX| = 1$. Find the point Y on side AC such that AX, BY, and CZ are concurrent. *[5]*



Solution. By Ceva's Theorem, AX, BY, and CZ will be concurrent exactly when $\frac{|AZ|\cdot|BX|\cdot|CY|}{|ZB|\cdot|XC|\cdot|YA|} = 1$. In this case we know that $|AZ| = 2$, $|ZB| = 3 - 2 = 1$, $|BX| = 1$, and $|XC| = 4 - 1 = 3$, so AX, BY, and CZ will be concurrent exactly when $\frac{2\cdot 1\cdot|CY|}{1\cdot 3\cdot|YA|} = 1$, *i.e.* when $\frac{|CY|}{|YA|} = \frac{3}{2}$. Since $|CY| + |YA| = b = 5$, it follows that AX, BY, and CZ will be concurrent exactly when $|CY| = 3$ and $|YA| = 2$. ■