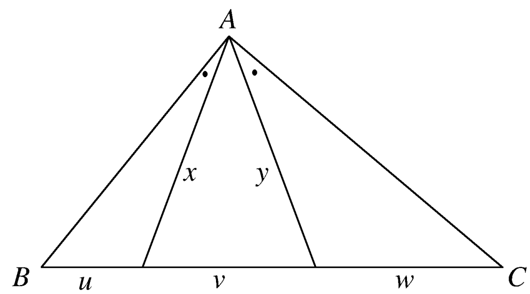
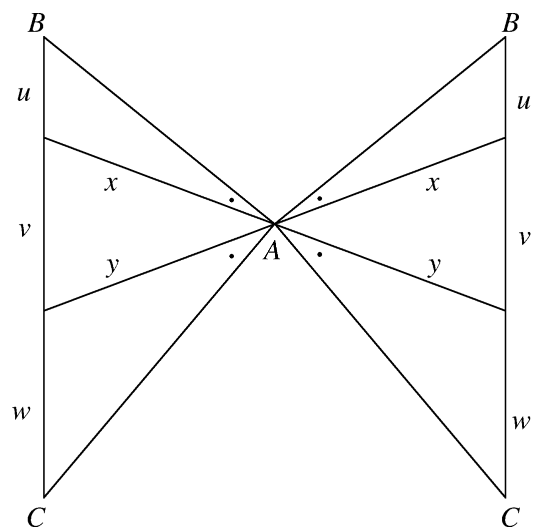
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #9

1. *(Exercise 3B.1)* Line segments of length x and y have been drawn from vertex A of $\triangle ABC$ to base BC, and these make equal angles with the sides, as indicated by the dots. These lines divide BC into segments of length u, v, and w, as shown. Find a formula for the ratio x/y in terms of the quantities u, v, and w. *[5]*



Solution. We will use Lemma 3.14, which was used to prove the Butterfly Theorem in Section 3B, to find the desired formula. To apply this lemma we need a configuration similar to the configuration in the Butterfly Theorem, which we get by putting together two copies of $\triangle ABC$ and the interior line segments, at vertex A so that the two copies of BC are parallel to each other.



Note that since the angles indicated by dots are equal, each of the interior line segments the two copies of $\triangle ABC$ forms a straight line with the copy of the other interior line segment, as do the sides AB from each copy of the triangle with AC from the other copy. We pick one of these straight lines, namely the one including the line segment of length x in the left copy of $\triangle ABC$ and the line segment of length y in the right copy of $\triangle ABC$. By Lemma 3.14 it follows that $\frac{x^2}{y^2} = \frac{u(v+w)}{(u+v)w}$, so $\frac{x}{y} = \sqrt{\frac{u(v+w)}{(u+v)w}}$ is a formula for the ratio x/y in terms of the quantities u, v, and w, as desired. ■

2. *(Exercise 3C.2)* Suppose that A, B, abd C are distinct and collinear and that X and Y lie on the line through them.

a. If $\mathbf{cr}(A,B,C,X) = \mathbf{cr}(A,B,C,Y)$, show that either X and Y are the same point, or else exactly one of A and B lies on the segment XY. *[3]*

b. If $\mathbf{cr}(A,B,C,X) = \mathbf{cr}(A,B,C,Y)$ and also $\mathbf{cr}(C,B,A,X) = \mathbf{cr}(C,B,A,Y)$, show that X and Y must be the same point. *[2]*

Solution to a. Assume $\mathbf{cr}(A,B,C,X) = \frac{|AC|\cdot|BX|}{|AX|\cdot|BC|} = \frac{|AC|\cdot|BY|}{|AY|\cdot|BC|} = \mathbf{cr}(A,B,C,Y)$. Note that this is true if and only if $\frac{|BX|}{|AX|} = \frac{|BY|}{|AY|}$, which, in turn, is true if and only if $|BX|\cdot|AY| = |BY|\cdot|AX|$.

If $X = Y$, we are done. If $X \neq Y$, we need to show that exactly one of A and B lies on the segment XY. Suppose, by way of contradiction, that this is not true, so either both or neither of A and B lies on the segment XY.

If both of A and B lie on the line segment XY, we may suppose that they are arranged in the order $XABY$. (The argument if they are arranged in the order $XBAY$ is very similar.) Then $|AX| < |BX|$ and $|AY| > |BY|$, so $|AX|\cdot|BY| < |BX|\cdot|AY|$, contradicting the assumption that $|BX|\cdot|AY| = |BY|\cdot|AX|$.

If neither of A and B lies on the segment AB, we may suppose that they are arranged in the order $AXYB$. (The argument if they are arranged in the order $AYXB$ is very similar.) Then $|AX| < |AY|$ and $|BX| > |BY|$, so $|AX|\cdot|BY| < |BX|\cdot|AY|$, contradicting the assumption that $|BX|\cdot|AY| = |BY|\cdot|AX|$.

Thus, if $X \neq Y$, exactly one of A and B must lie on the segment XY, as desired. ■

Solution to b. Assume that $\mathbf{cr}(A,B,C,X) = \mathbf{cr}(A,B,C,Y)$ and $\mathbf{cr}(C,B,A,X) = \mathbf{cr}(C,B,A,Y)$. As noted in the solution to part **a** above, the former is true exactly when $|BX|\cdot|AY| = |BY|\cdot|AX|$. Similarly, the latter is true exactly when $|BX|\cdot|CY| = |BY|\cdot|CX|$. It follows from these that $\frac{|AX|}{|AY|} = \frac{|BX|}{|BY|} = \frac{|CX|}{|CY|}$, so $|AX|\cdot|CY| = |CX|\cdot|AY|$, and hence we also have that

$$\mathbf{cr}(A,C,B,X) = \frac{|AB|\cdot|CX|}{|AX|\cdot|CB|} = \frac{|AB|\cdot|CY|}{|AY|\cdot|CB|} = \mathbf{cr}(A,C,B,Y)\,.$$

By part **a**, it follows that if $X \neq Y$, then exactly one of A and B lies on the segment XY, exactly one of B and C lies on the segment XY, and exactly one of A and C lies on the segment XY. Thie first two imply that B must be on the segment XY and both A and C cannot be, but the third contradicts this, requiring that (exactly) one of A and C be on the segment XY. Thus, by contradiction, it must be the case that $X = Y$. ■