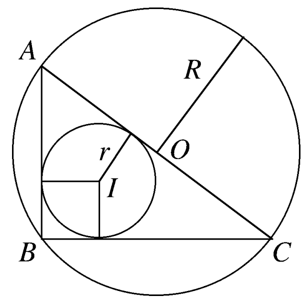
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #7

1. *(Exercise 2E.3)* Show that in a right triangle, the inradiius, circumradius, and semiperimeter are related by the formula $s = r + 2R$. *[5]*



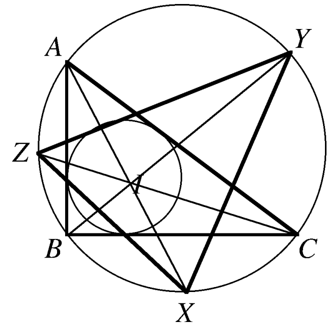
Solution. Suppose $\triangle ABC$ is a right triangle such that $\angle B = 90°$. It follows that the circumcentre O of $\triangle ABC$ is the middle point of AC (see the discussion on pages 50-51), and so $b = |AC| = 2R$.

Let I be the incentre of $\triangle ABC$. Let U on BC, V on AC, and W on AB be the points at which the incircle is tangent to the sides of $\triangle ABC$. Then $IU \perp BC$ and $IW \perp AB$, since IU and IW are radii of the incircle, and since $\angle B = 90°$, it follows that $WBUI$ is a square and $|BW| = |BU| = r$.

By Lemma 2.27, we also have $|BW| = |BU| = s - b$. Plugging in the relationships noted above gives us $r = |BW| = s - b = s - 2R$, from which it follows that $s = r + 2R$, as desired. ■

2. *(Exercise 2E.5)* Extend the bisectors of $\angle A$, $\angle B$, and $\angle C$ of $\triangle ABC$ to meet the circumcircle at points X, Y, and Z. Show that I is the orthocentre of $\triangle XYZ$. *[5]*

Hint: Show that XY is the perpendicular bisector of IC.



Solution. Following the hint, let P be the point of intersection of XY and IC. By Lemma 2.32 we have that $|IY| = |CY|$ and $|IX| = |CX|$. Since we also have $|XY| = |XY|$, $\triangle IYX \cong \triangle CYX$ by the side-side-side congruence test, and hence $\angle IYP = \angle IYX =$

$\angle CYX = \angle CYP$. As we also have $|IY| = |CY|$ and $|YP| = |YP|$, $\triangle IYP \cong \triangle CYP$ by the side-angle-side congruence test. This tells us that $|IP| = |CP|$, so XY bisects IC. It also tells us that $\angle IPY = \angle CPY$, and since these angles add up to a straight angle, it follows that $\angle IPY$ and $\angle CPY$ are right angles. Thus XY is a perpendicular bisector of IC, and hence is perpendicular to the extension ZC of IP too.

Similar arguments show that XA is perpendicular to YZ and that BY is perpendicular to XZ. It follows that XA, BY, and CZ are the altitudes of $\triangle XYZ$, so their common point of intersection, the incentre I of $\triangle ABC$, is the orthocentre of $\triangle XYZ$. ■