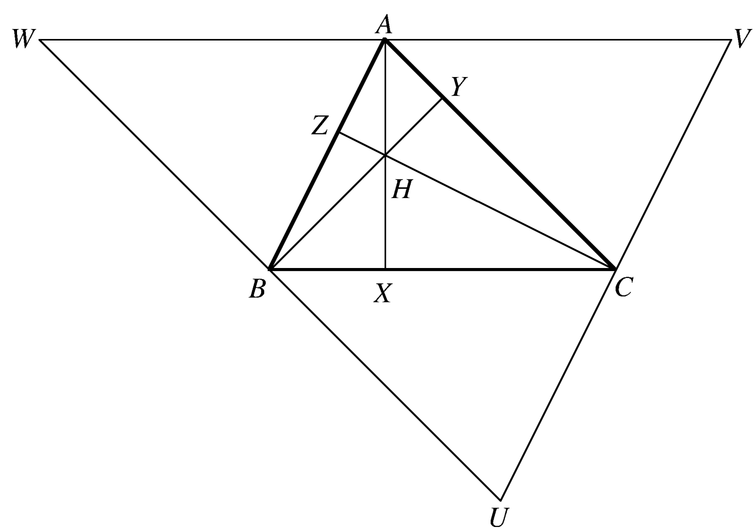
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #6

1. *(Exercise 2C.3)* Given $\triangle ABC$, draw line WV through A parallel to BC, line UW through B parallel to AC, and line UV through C parallel to AB. Show that the orthocentre of $\triangle ABC$ is the circumcentre of $\triangle UVW$. *[5]*

Solution. Given $\triangle ABC$, draw $\triangle UVW$ as described in the problem. Let AX, BY, and CZ be the altitudes of $\triangle ABC$, and let H be its orthocentre.



Note that AX is perpendicular to VW because VW is parallel to BC, BY is perpendicular to UW because UW is parallel to AC, and CZ is parallel to UV because UV is parallel to AB. Since the circumcentre of $\triangle UVW$ is the point where the perpendicular bisectors of its sides meet, it follows that all we need to show that H is the circumcentre of $\triangle UVW$ is to check that A, B, and C are the midpoints of the sides of $\triangle UVW$.

Since AB is a transversal between the parallel lines UW and AC, we have that $\angle BAC = \angle ABW$. Similarly, since AB is a transversal between the parallel lines VW and BC, we also have that $\angle CBA = \angle BAW$. As $|AB| = |AB|$, it follows by the side-angle-side congruence criterion that $\triangle ABC \cong \triangle BAW$, and so $|AW| = |BC|$.

Since AC is a transversal between the parallel lines UV and AB, we have that $\angle BAC = \angle ACV$. Similarly, since AC is a transversal between the parallel lines VW and BC, we also have that $\angle ACB = \angle VAC$. As $|AC| = |AC|$, it follows by the side-angle-side congruence criterion that $\triangle ABC \cong \triangle CVA$, and so $|AV| = |BC|$.

Thus $|AW| = |BC| = |AV|$, so A is the midpoint of VW. Similar arguments show that B is the midpoint of UW and C is the midpoint of UV. Hence the altitudes AX, BY, and CZ of $\triangle ABC$ are also the perpendicular bisectors of the sides of $\triangle UVW$, so their intersection, the orthocentre H of $\triangle ABC$, is also the circumcentre of $\triangle UVW$. ■

2. *(Exercise 2C.4)* Show that the nine-point circle of $\triangle ABC$ is the locus of all midpoints of segments UH, where H is the orthocentre of $\triangle ABC$ and U is an arbitrary point of the circumcircle. *[5]*

Hint: By Exercise 1H.10, we already know that this locus is a circle.

Solution. Following the hint, we first review what it refers to:

1H.10 Given a circle centered on a point O and an arbitrary point P, consider the locus of all points Y that occur as midpoints of segments PX, where X lies on the given circle. Show that this locus is a circle with radius half that of the original circle. Locate the center of the locus.

It follows from this that the locus of all midpoints of segments UH, where H is the orthocentre of $\triangle ABC$ and U is an arbitrary point of the circumcircle, is indeed a circle. Since any three distinct points on a circle uniquely determine that circle (see Theorem 1.15), all we have to do to show that the locus is the nine-point circle of $\triangle ABC$ is to show that at least three distinct points of the locus are on the nine-point circle.

Note that A, B, and C are on the circumcircle of $\triangle ABC$. Then the midpoints of AH, BH, and CH are on the locus in question, by the definition of that locus. These mispoints are also the three Euler points of $\triangle ABC$, by the definition of Euler points on pp. 62-63, and hence are on the nine-point circle by Theorem 2.12. Since they have three points in common, the two circles are the same circle. ■