Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

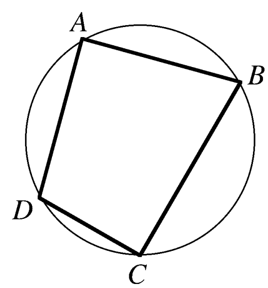
Solutions to Problem Set #5

1. *(Exercise 2A.1)* Show that quadrilateral $ABCD$ can be inscribed in a circle if and only if $\angle B$ and $\angle D$ are supplementary. *[5]*

Hint: To prove if, show that D lies on the unique circle through A, B, and C.

Solution. We will prove the two directions of the "if and only if" argument separately.

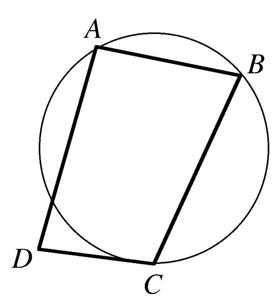
($\Longrightarrow$) Assume that quadrilateral $ABCD$ is cyclic, *i.e.* it can be inscribed in a circle.

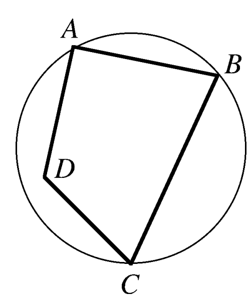


By Corollary 1.17 in the text, opposite angles of an inscribed quadrlateral, such as $\angle B$ and $\angle D$ are here, are supplementary. □

($\Longleftarrow$) Assume that $\angle B$ and $\angle D$ of quadrilateral $ABCD$ are supplementary. We will show, using an argument by contradiction, that D must lie on the unique circle defined by A, B, and C.

Suppose then, by way of contradiction, that D does *not* lie on the unique circle defined by A, B, and C. Then there are two possibilities: either D lies outside the circle or D lies inside the circle.





If D lies outside the circle, let X and Y be the points in which DA and DC intersect the circle. (It is possible that X or Y, or both, may be equal to B or C, respectively.) Corollary 1.18 then tells us that $\angle D$ is equal in degrees to $\frac{1}{2}\left(\text{arc}(ABC) - \text{arc}(XY)\right)$. Theorem 1.16 tells us that $\angle B$ is equal in degrees to $\frac{1}{2}\text{arc}(AXC)$. Since $\text{arc}(ABC) + \text{arc}(AXC) = 360°$, it follows that:

$$\angle B + \angle D = \frac{1}{2}\text{arc}(AXC) + \frac{1}{2}\left(\text{arc}(ABC) - \text{arc}(XY)\right) = \frac{1}{2}360° - \frac{1}{2}\text{arc}(XY) < 180°$$

Similarly, if D lies inside the circle, let X and Y be the points in which DA and DC intersect the circle. (Can X or Y be equal to B or C, respectively?) Corollary 1.19 then tells us that $\angle D$ is equal in degrees to $\frac{1}{2}(\text{arc}(ABC) + \text{arc}(XY))$. Theorem 1.16 tells us that $\angle B$ is equal in degrees to $\frac{1}{2}\text{arc}(AXC)$. Since $\text{arc}(ABC)+\text{arc}(AXC) = 360°$, it follows that:

$$\angle B + \angle D = \frac{1}{2}\text{arc}(AXC) + \frac{1}{2}(\text{arc}(ABC) + \text{arc}(XY)) = \frac{1}{2}360° + \frac{1}{2}\text{arc}(XY) > 180°$$

Either way, if D is not on the circle defined by A, B, and C, then $\angle B + \angle D \neq 180°$, *i.e.* they are not supplementary, a contradiction to the fact that they are. Hence, D must be on the circle defined by A, B, and C. □

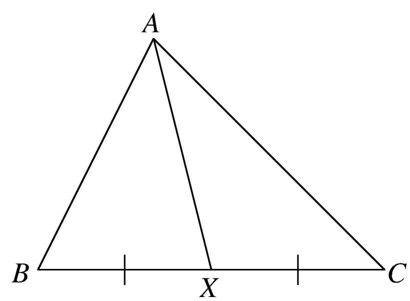
This completes the proof. ■

2. *(Exercise 2B.3)* Show that there can be no point P in $\triangle ABC$ such that every line through P subdivides the triangle into two pieces of equal area. *[5]*

Hint: If there is a point that has this property, show that the medians would have to pass through P.

Solution. Following the hint, we first show that the medians of $\triangle ABC$ would have to pass through any point P such that every line through P subdivides the triangle into two pieces of equal area. Assume, for the sake of argument, that such a point P existed.

Suppose the median from A met BC in point X; by the definition of median this means that $|BX| = |XC|$.

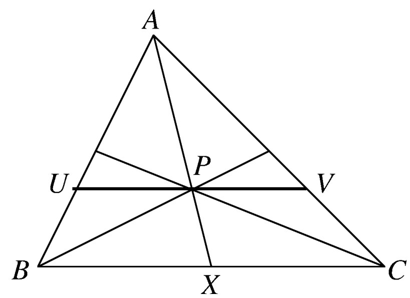


Then $\triangle ABX$ and $\triangle AXC$ have the same height and bases – BX and XC, respectively – of equal length, so they must have the same area. This means that the median AX subdivides $\triangle ABC$ into pieces of equal area, which means that AX would have to pass through P. Similar arguments show that the other medians of $\triangle ABC$ would have to pass through P as well.

Since the medians of $\triangle ABC$ meet in a common point, namely the centroid of the triangle, P would have to be the centroid of $\triangle ABC$. To show that there is no such P it now suffices to find a line through the centroid that does *not* divide $\triangle ABC$ into pieces of equal area.

Suppose points U on side AB and V on side BC are chosen so that UV passes through the centroid – let's call it P! – of $\triangle ABC$ and $UV \parallel BC$.

Since $UV \parallel BC$ and AB and AC are transversals, we have that $\angle AUP = \angle ABX$ and $\angle AVP = \angle ACX$. We also have that $\angle UAP = \angle BAX$ (these are the same angle)



and $\angle VAP = \angle CAX$ (and so are these). It now follows that $\triangle AUP \sim \triangle ABX$ and $\triangle AVP \sim \triangle ACX$, in each case by the angle-angle similarity criterion. Since P is the centroid, $\frac{|AP|}{|AX|} = \frac{2}{3}$, and it then follows from the similarities noted above that:

$$\frac{|AU|}{|AB|} = \frac{|UP|}{|BX|} = \frac{|AP|}{|AX|} = \frac{2}{3} = \frac{|PV|}{|XC|} = \frac{|AV|}{|AC|}$$

Since $|UV| = |UP| + |PV|$ and $|BC| = |BX| + |XC|$, it also follows that:

$$\frac{|UV|}{|BC|} = \frac{|AU|}{|AB|} = \frac{|AV|}{|AC|} = \frac{2}{3}$$

Hence $\triangle AUV \sim \triangle ABC$ by the side-side-side similarity criterion, with a scale factor of $\frac{2}{3}$. It follows that the area of $\triangle AUV$ is $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ of the area of $\triangle ABC$ (see the discussion on p. 48 of the text). Since $\frac{4}{9} \neq \frac{1}{2}$, UV is a line through P which does not divide $\triangle ABC$ into pieces of equal areas, as desired. ■