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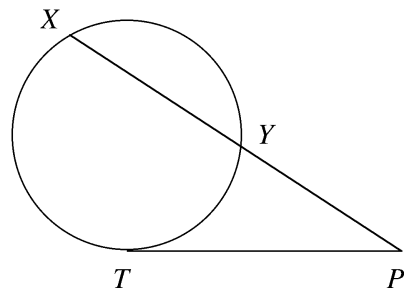
Mathematics 226H – Geometry I: Euclidean geometry
Trent University, Winter 2008

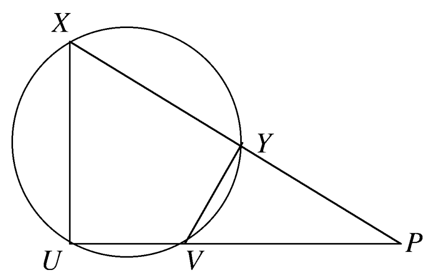
Solutions to Problem Set #4

1. *(Exercise 1H.5)* Given two points X and Y on a circle, a point P is chosen on line XY, outside of the circle, and tangent PT is drawn, where T lies on the circle. Show that $|PX| \cdot |PY| = |PT|^2$. *[5]*



Solution. We'll actually prove a more general fact:

Suppose X, Y, U, and V are points on a circle such that XY and UV meet in a point P outside the circle. Then $|PX| \cdot |PY| = |PU| \cdot |PV|$.



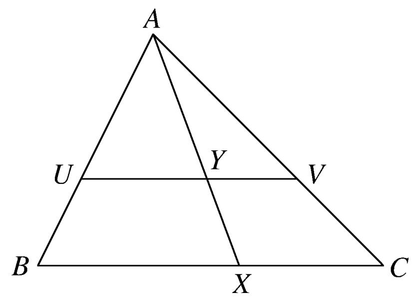
This is a restatement of half of Theorem 1.35 in the text. The given problem is the special case where the points U and V coincide and the chord UV is instead a tangent line.

Since $XYVU$ is a cyclic quadrilateral, $\angle UXY + \angle UVY = 180°$, and since the points U, V, and P are collinear, $\angle UVY + \angle YVP = 180°$. It follows that $\angle UXY = \angle YVP$.

Since we also have $\angle VYP = \angle XPU$ (because they're the same angle), it now follows by angle-angle similarity that $\triangle UXP \sim \triangle YVP$. This means that $\frac{|PX|}{|PV|} = \frac{|PU|}{|PY|}$, from which it follows that $|PX| \cdot |PY| = |PU| \cdot |PV|$, as desired. ∎

Note: There are direct ways to prove the problem, but this was just too cute to resist. Note that the argument above is somewhat different from the proof of Theorem 1.35 given in the text. The paranoid and eagle-eyed may still wonder, of course, whether and exactly how the given problem follows from the more general fact proven above.

2. *(Exercise 1H.8)* Let X be a point on side BC of $\triangle ABC$ and draw AX. Let U and V be points on side AB and AC, respectively, chosen so that $UV \parallel BC$, and let Y be the point where AX cuts UV. Show that $\frac{|UY|}{|YV|} = \frac{|BX|}{|XC|}$. *[5]*



Solution. $\angle UAY = \angle BAX$ because these are the same angle, and $\angle AUY = \angle ABX$ because $UV \parallel BC$ and AB is a transversal. Thus $\triangle AUY \sim \triangle ABX$ by angle-angle similarity, and so $\frac{|UY|}{|BX|} = \frac{|AY|}{|AX|}$.

Similarly, $\angle VAY = \angle CAX$ because these are the same angle, and $\angle AVY = \angle ACX$ because $UV \parallel BC$ and AC is a transversal. Thus $\triangle AYV \sim \triangle AXC$ by angle-angle similarity, and so $\frac{|AY|}{|AX|} = \frac{|YV|}{|XC|}$.

Putting the above together, we get

$$\frac{|UY|}{|BX|} = \frac{|AY|}{|AX|} = \frac{|YV|}{|XC|} \quad \Longrightarrow \quad |UY| \cdot |XC| = |YV| \cdot |BX| \quad \Longrightarrow \quad \frac{|UY|}{|YV|} = \frac{|BX|}{|XC|},$$

as desired. ■