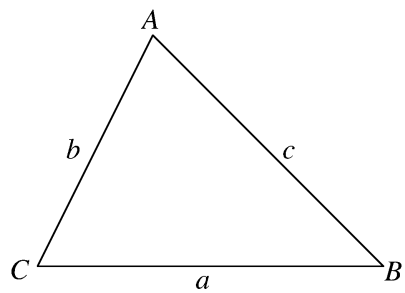
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

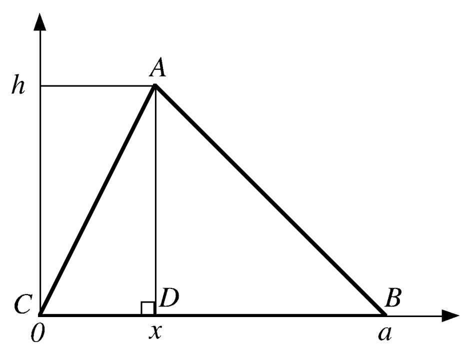
Solutions to Problem Set #3

1. *(Exercise 1E.3)* Since a triangle is determined by angle-side-angle, there should be a formula for K_{ABC} expressed in terms of a and $\angle B$ and $\angle C$. Derive such a formula. *[5]*



Solution. We'll derive the desired formula in as brutally low-power a way as we can, using the basic area formula for a triangle, $\frac{1}{2}$base $\times$ height, some algebra using Cartesian coordinates, and a little trigonometry. Since we are given the base, $a = |BC|$, our main objective will be to express the height, h, of the triangle in terms of a, $\angle B$, and $\angle C$.

First, we put $\triangle ABC$ down on the Cartesian plane so that B is at the origin, C is at $(a, 0)$, and A is above the x-axis at the point (x, h). We'll denote the point where the altitude from A meets BC by D, which has coordinates $(x, 0)$.



Note that h is the height of the triangle if BC is used as the base, so the area of the triangle is $\frac{1}{2}|BC|h = \frac{1}{2}ah$.

Second, considering the right triangle $\triangle ABD$, we get $\tan(\angle B) = h/x$. Note that this is the slope of the line that AB is part of. Similarly, considering the right triangle $\triangle ACD$, we get $\tan(\angle C) = h/(a-x)$. Note that the slope of the line that AC is part of is the negative of this, namely $-\tan(\angle C) = -h/(a-x)$.

Third, the equation of the line passing through $B = (0, 0)$ with slope $\tan(\angle B)$ is $y = \tan(\angle B)x$, and the equation of the line passing through $C = (a, 0)$ with slope $-\tan(\angle C)$ is $y = -\tan(\angle C)(x-a) = -\tan(\angle C)x + \tan(\angle C)a$.

Fourth, we find the coordinates of $A = (x, h)$, the point where the two lines intersect, in terms of a, $\tan(\angle B)$, and $\tan(\angle C)$, by comparing the equations of the lines and solving for x and then h. Solve for x:

$$\begin{aligned} & \tan(\angle B)x = -\tan(\angle C)x + \tan(\angle C)a \\ \implies \quad & (\tan(\angle B) + \tan(\angle C))\, x = \tan(\angle C)a \\ \implies \quad & x = \frac{a\tan(\angle C)}{\tan(\angle B) + \tan(\angle C)} \end{aligned}$$

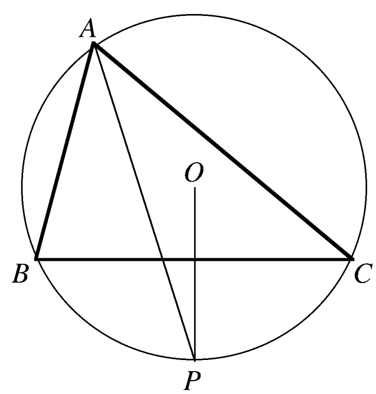
Now we can get h by plugging the value for x into the equation of one of the lines:

$$h = \tan(\angle B)x = \frac{a\tan(\angle B)\tan(\angle C)}{\tan(\angle B) + \tan(\angle C)}$$

Finally, we plug our value for h into the basic formula for the area of a triangle to get the formula we want:

$$\begin{aligned} K_{ABC} &= \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}ah \\ &= \frac{1}{2}a \cdot \frac{a\tan(\angle B)\tan(\angle C)}{\tan(\angle B) + \tan(\angle C)} \\ &= \frac{a^2}{2} \cdot \frac{\tan(\angle B)\tan(\angle C)}{\tan(\angle B) + \tan(\angle C)} \qquad \blacksquare \end{aligned}$$

2. *(Exercise 1F.9)* In Figure 1.33 [reproduced below], point O is the centre of the circumcircle of $\triangle ABC$, and the bisector of $\angle A$ is extended to meet the circle at P. Prove that the radius OP is perpendicular to BC. *[5]*



Solution. Let Q be the point of intersection of BC and OP. Now

$$\begin{aligned} \angle BOQ &= \angle BOP && \text{Because they're the same angle.} \\ &= 2\angle BAP && \text{By Theorem 1.16.} \\ &= \angle A && \text{Because } AP \text{ is the angle bisector of } \angle A. \end{aligned}$$

and, similarly,

$$\begin{aligned} \angle COQ &= \angle COP && \text{Because they're the same angle.} \\ &= 2\angle CAP && \text{By Theorem 1.16.} \\ &= \angle A && \text{Because } AP \text{ is the angle bisector of } \angle A. \end{aligned}$$

It follows that $\angle BOQ = \angle COQ$. Since we also have $|OB| = |OC|$ (as both are radii of the circumcircle) and $|OQ| = |OQ|$ (duh!), the SAS congruence criterion tells us that $\triangle OBQ \cong \triangle OCQ$. Hence $\angle OQB = \angle OQC$; since $\angle OQB + \angle OQC = 180°$, it follows that $\angle OQB = \angle OQC = \frac{1}{2}180° = 90°$. Thus OP is perpendicular to BC. ■