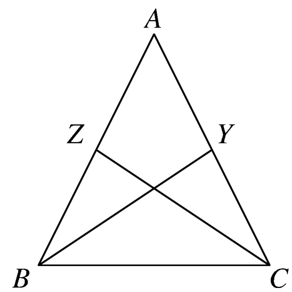
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #2

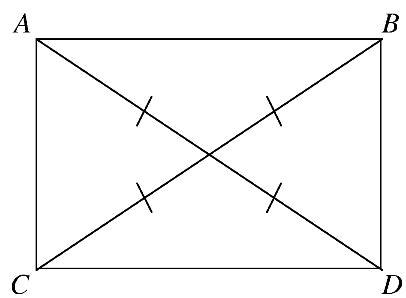
1. *(Exercise 1B.7)* Suppose BY and CZ are altitudes of $\triangle ABC$ and $|BY| = |CZ|$. Show that $|AB| = |AC|$. *[5]*



Solution 1. Since we have $|BY| = |CZ|$ (given), $\angle AZC = 90° = \angle AYB$ (as BY and CZ are altitudes), and $\angle ZAC = \angle BAC = \angle YAB$, it follows by the SAA congruence criterion that $\triangle AZC \cong \triangle AYB$. Hence, since corresponding sides of congruent triangles must be congruent, $|AB| = |AC|$, *i.e.* $\triangle ABC$ is isosceles. ∎

Solution 2. Although one is not supposed to use ideas from later in the text to do a given problem, it is very tempting to do so here. Note that $\frac{1}{2}|AB| \cdot |CZ| = K_{ABC} = \frac{1}{2}|AC| \cdot |BY|$. Since $|BY| = |CZ|$, it follows that $|AB| = |AC|$. ∎

2. *(Exercise 1D.5)* Prove that a parallelogram with equal diagonals is a rectangle. *[5]*



Solution. Call the point of intersection of the diagonals of the parallelogram P. By Theorem 1.9, P must be the midpoint of each diagonal and, since the diagonals are equal in length, we must have $|AP| = |BP| = |CP| = |DP|$. By Theorem 1.6, we also have that $|AB| = |CD|$ and $|AD| = |BC|$. It follows by the SSS congruence criterion that $\triangle ABP \cong \triangle CDP$ and $\triangle ADP \cong \triangle BCP$. Since these triangles are all isosceles, we get that

$$\angle PAB = \angle ABP = \angle PCD = \angle PDC \quad \text{and} \quad \angle PAD = \angle PDA = \angle PBC = \angle PBC\,.$$

Observe that each of the four internal angles of the parallelogram $ABCD$ is the sum of one of the first four equal angles and one of the second four equal angles above. Hence each of the four internal angles of $ABCD$ is equal; since their sum must be $180(4-2) = 360°$ by Problem 1.5, each angle is $\frac{1}{4}360 = 90°$, *i.e.* a right angle. Thus $ABCD$ is a rectangle. ∎