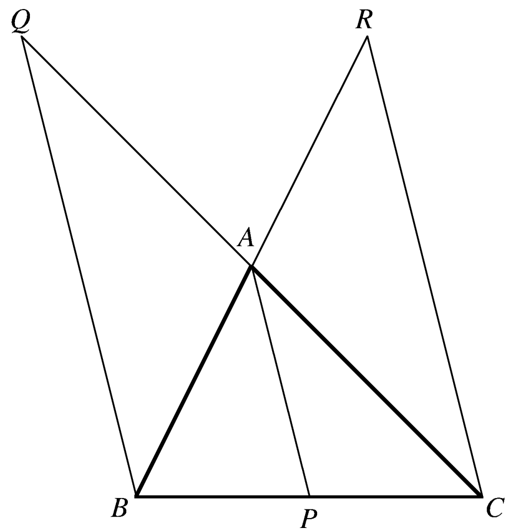
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

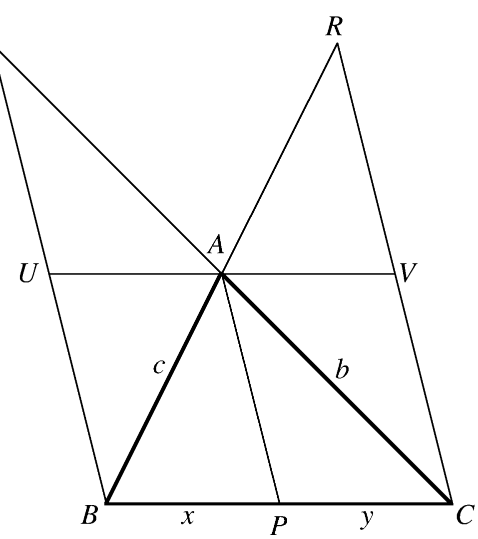
Problem Set #11
Due on Friday, 4 April, 2008.

1. *(Exercise 4B.2)* Show using similar triangles that if Cevians AP, BQ, and CR are parallel, then the Cevian product is trivial. *[5]*



Hint: Let a, b, and c denote the lengths of the sides of $\triangle ABC$ and write $x = |BP|$ and $y = |PC|$. Express each of the six lengths that appear in the Cevian product in terms of the five quantities a, b, c, x, and y.

Solution. Following the hint, let a, b, and c denote the lengths of the sides of $\triangle ABC$ and write $x = |BP|$ and $y = |PC|$. The Cevian product we need to consider is $\frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|}$. To follow the hint, observe that we still need to write $|AR|$, $|RB|$, $|CQ|$, and $|QA|$ in terms of a, b, c, x, and y. Draw a line through A parallel to BC meeting BR at U and CQ at V:



Since $UA \parallel BP$ and $BU \parallel PA$, $UABP$ is a parallelogram, and so $|UA| = |BP| = x$ and $|BU| = |PA|$. Similarly, because $AV \parallel PC$ and $PA \parallel CV$, $AVCP$ is a paralellogram, and so $|AV| = |PC| = y$ and $|CV| = |PA|$.

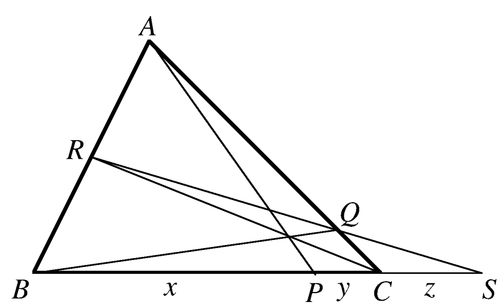
Since $QU \parallel AP$ and $UA \parallel PC$, $\angle QUA = \angle APC$, and since QC is a transversal between the parallel lines QA and PC, we also have $\angle QAU = \angle ACP$. Hence $\triangle QUA \sim \triangle APC$, and so $\frac{|QA|}{b} = \frac{|QA|}{|AC|} = \frac{|UA|}{|PC|} = \frac{x}{y}$, which gives us that $|QA| = bx/y$. A similar argument shows that $\triangle RAV \sim \triangle ABP$ and $|AR| = cy/x$. It follows from these that $|CQ| = |CA| + |AQ| = b + bx/y = (by + bx)/y = b(y + x)/y = ba/y$ and $|BR| = |BA| + |AR| = c + cy/x = (cx + cy)/x = c(x + y)/x = ca/x$.

We can now compute the Cevian product in terms of a, b, c, x, and y:

$$\frac{|AR|}{|BR|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = \frac{cy/x}{ca/x} \cdot \frac{x}{y} \cdot \frac{ba/y}{bx/y} = \frac{y}{a} \cdot \frac{x}{y} \cdot \frac{a}{x} = \frac{yxa}{ayx} = 1$$

Thus the Cevian product is trivial if AP, BQ, and CR are parallel, as desired. ■

2. *(Exercise 4D.3)* Three concurrent Cevians AP, BQ, and CR are drawn in $\triangle ABC$, as shown in the figure, and RQ is extended to meet the extension of BC at S. Apply both Ceva's and Menelaus' theorem in $\triangle ABC$ to prove that $xz = y(x + y + z)$, where we have written $|BP| = x$, $|PC| = y$, and $|CS| = z$, as indicated. *[5]*



Solution. Since the Cevians AP, BQ, and CR are concurrent, Ceva's Theorem tells us that

$$\frac{|AR|}{|BR|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = \frac{|AR|}{|BR|} \cdot \frac{x}{y} \cdot \frac{|CQ|}{|QA|} = 1\,.$$

Also, since R, Q, and S are collinear, and $|BS| = |BP| + |PC| + |CS| = x + y + z$, Menelaus' Theorem tells us that

$$\frac{|AR|}{|BR|} \cdot \frac{|BS|}{|SC|} \cdot \frac{|CQ|}{|QA|} = \frac{|AR|}{|BR|} \cdot \frac{x + y + z}{z} \cdot \frac{|CQ|}{|QA|} = 1\,.$$

Rearranging and comparing these equations gives us that

$$\frac{x}{y} = \frac{|BR|}{|AR|} \cdot \frac{|QA|}{|CQ|} = \frac{x + y + z}{z}\,,$$

and cross-multiplying this gives $xz = y(x + y + z)$, as desired. ■