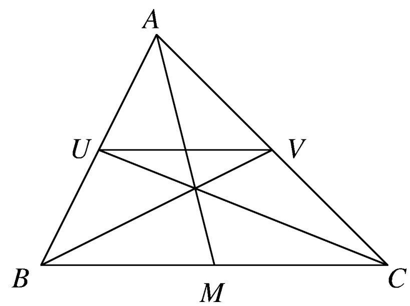
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #10

1. *(Exercise 4A.2)* Let U and V be points on sides AB and AC, respectively, of $\triangle ABC$ and suppose that UV is parallel to BC. Show that the intersection of UC and VB lies on the median AM. *[5]*



Solution. Since $UV \parallel BC$, $\angle AUV = \angle ABC$ and $\angle AVU = \angle ACB$. It follows that $\triangle AUV \sim \triangle ABC$, and so $\frac{|AU|}{|AB|} = \frac{|AV|}{|AC|}$. Since $|AB| = |AU|+|UB|$ and $|AC| = |AV|+|VC|$, we also get that

$$\frac{|UB|}{|AU|} + 1 = \frac{|UB| + |AU|}{|AU|} = \frac{|AB|}{|AU|} = \frac{|AC|}{|AV|} = \frac{|VC| + |AV|}{|AV|} = \frac{|VC|}{|AV|} + 1\,.$$

Thus $\frac{|UB|}{|AU|} = \frac{|VC|}{|AV|}$, and hence $\frac{|AU|}{|UB|} = \frac{|AV|}{|VC|}$.

Note also that if f AM is a median of $\triangle ABC$, then, by definition, $|BM| = |MC|$, so $\frac{|BM|}{|MC|} = 1$.

Computing the Cevian product for the Cevians AM, BV, and CU we get, using the relations obtained above:

$$\frac{|AU|}{|UB|} \cdot \frac{|BM|}{|MC|} \cdot \frac{|CV|}{|VA|} = \frac{|AV|}{|VC|} \cdot 1 \cdot \frac{|CV|}{|VA|} = 1$$

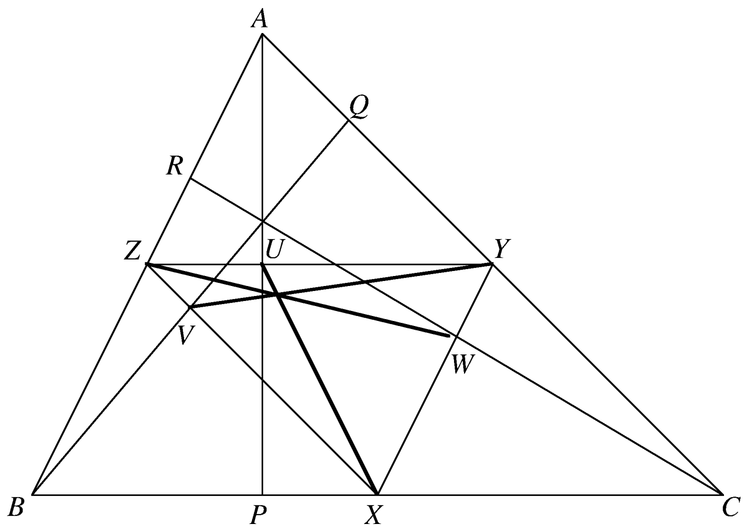
By Ceva's Theorem, it follows that AM, BV, and CU are concurrent, as desired. ∎

2. *(Exercise 4A.5)* Given three concurrent Cevians in a triangle, show that the three lines obtained by joining the midpoints of the Cevians to the midpoints of the corresponding sides are concurrent. *[5]*

Hint: Consider the medial triangle.

Solution. Suppose AP, BQ, and CR are concurrent Cevians of $\triangle ABC$, respectively, and let X, Y, and Z be the midpoints of sides BC, AC, and AB, respectively. $\triangle XYZ$ is the medial triangle of $\triangle ABC$, and it follows from Lemma 1.29 (among other results) that $XY \parallel AB$, $YZ \parallel BC$, and $XZ \parallel AC$. In turn, this implies that $\triangle ABC \sim \triangle XYZ$.

Let U be the intersection of AP with YZ, V be the intersection of BQ with XZ, and W be the intersection of CR with XY.



We claim that U is the midpoint of AB. Since AB and AP are transversals meeting the parallel lines YZ and BC, $\angle AZU = \angle ABP$ and $\angle AUZ = APB$. Hence $\triangle AZU \sim \triangle ABP$ by the angle-angle similarity criterion. It follows that $\frac{|AU|}{|AP|} = \frac{|AZ|}{|AB|} = \frac{1}{2}$ (recall that Z is the midpoint of AB), and hence U is the midpoint of AB. Similar arguments show that V and W are the midpoints of BQ and CR, respectively.

Note that it is a consequence of the argument in the previous paragraph that $|UZ| = \frac{1}{2}|BP|$. Similar arguments can also be used to show that $|YU| = \frac{1}{2}|PC|$, $|XW| = \frac{1}{2}|BR|$, $|WY| = \frac{1}{2}|RA|$, $|VX| = \frac{1}{2}|CQ|$, and $|ZV| = \frac{1}{2}|QA|$. Observe that XU, YV, and ZW are Cevians of $\triangle XYZ$. By Ceva's Theorem they will be concurrent if and only if the corresponding Cevian product equals 1:

$$\begin{aligned}\frac{|XW|}{|WY|} \cdot \frac{|YU|}{|UZ|} \cdot \frac{|ZV|}{|VX|} &= \frac{\frac{1}{2}|BR|}{\frac{1}{2}|RA|} \cdot \frac{\frac{1}{2}|PC|}{\frac{1}{2}|BP|} \cdot \frac{\frac{1}{2}|QA|}{\frac{1}{2}|CQ|} = \frac{|BR|}{|RA|} \cdot \frac{|PC|}{|BP|} \cdot \frac{|QA|}{|CQ|} \\ &= \frac{|BR|}{|RA|} \cdot \frac{|AQ|}{|QC|} \cdot \frac{|CP|}{|PB|} = 1 \\ &\qquad \text{(Since the given Cevians of } \triangle ABC \text{ are concurrent.)}\end{aligned}$$

Thus XU, YV, and ZW are concurrent, as desired. ■