Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Winter 2008

Solutions to Problem Set #1

1. Go through Euclid's proof of Proposition I-1 in the *Elements* and identify at each step the use, implicit or explicit, of his definitions, postulates, and/or common notions. *[5]*

Solution. Proposition I-1 in the *Elements* reads:

I-1. To construct an equilateral triangle on a given finite straight-line.

We will reproduce the given proof, adding annotations indicating which of Euclid's definitions, postulates, and/or common notions are being used at each step. The translator's annotations to this effect will remain enclosed in square brackets [], while additional ones will be enclosed in curly brackets {}.

> Let AB be the given finite straight-line {Definition 4}.
>
> So it is required to construct an equilateral triangle {Definition 20} on the straight-line AB {Definition 4}.
>
> Let the circle BCD {Definition 15} with center A {Definition 16} and radius AB {implicit in Definition 15} have been drawn [Postulate 3], and again let the circle ACE {Definition 15} with center B {Definition 16} and radius BA {implicit in Definition 15} have been drawn [Postulate 3]. And let the straight-lines CA and CB {Definition 4} have been joined from the point C {Definition 1}, where the circles {Definition 15} cut one another, to the points A and B {Definition 1} (respectively) [Postulate 1]. And since the point A {Definition 1} is the center {Definition 16} of the circle CDB {Definition 15}, AC is equal to AB [Definition 15]. Again, since the point B {Definition 1} is the center {Definition 16} of the circle CAE {Definition 15}, BC is equal to BA [Definition 15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [Common Notion 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) {Definition 4} CA, AB, and BC are equal to one another. Thus, the triangle ABC {Definitions 19 & 20} is equilateral {Definition 20}, and has been constructed on the given finite straight-line AB {Definition 4}. (Which is) the very thing it was required to do.

The above annotations are, frankly, overkill: after noting once, say, that a point is defined in Definition 1, there is really very little point in doing so again and again. It is worth noting that it could have been worse: various definitions use previous definitions and these previous ones are mostly not noted above. ■

2. Try to prove, as completely as you can, Proposition I-1 in the *Elements* from Hilbert's axioms for geometry. *[5]*

Solution. There are two possible approaches here: try to justify Euclid's proof of Proposition I-1 using Hilbert's system of axioms or try to devise (or look up) a different proof of Proposition I-1 using Hilbert's system of axioms. The former approach is harder than it looks because Euclid's system deals with circles directly, whereas Hilbert's does not, while the latter approach requires one to come up with a new proof using a less-intuitive system of axioms. We will go with the latter approach here.

There is a discussion, with a number of examples, about doing geometrical constructions using Hilbert's system, in Chapter VII of Hilbert's *Foundations of Geometry*. Some of the tricks – er, techniques – used below come from that chapter. It also includes some passing comments suggesting why justifying Euclid's proof using Hilbert's axioms would likely be quite difficult.

The key to the proof of Proposition I-1 given below is to construct an angle of $60° = \pi/3$ rad. We will prove a sequence of results below culminating in the construction of such an angle and then use it to prove Euclid's Proposition I-1. In what follows (starting with Lemma 2) we will assume that we have a line segment of length 1 available for reference; we can use the line segment given in the hypotheses of Proposition I-1 for this purpose. (In effect, this line segment is used to provide a standard unit for measuring distances.)

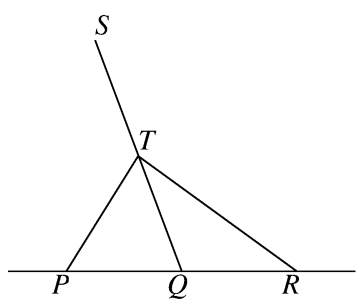
Lemma 0. Suppose that $\triangle ABC$ is isosceles with $|AB| = |AC|$. Then $\angle ABC = \angle ACB$.

Proof. Since $|AB| = |AC|$, $\angle BAC = \angle CAB$, and $|AC| = |AB|$, it follows by Axiom IV.6 in Hilbert's system that $\angle ABC = \angle ACB$. □

Note that Axiom IV.6 is essentially the SAS congruence criterion.

Lemma 1. One can construct a right angle.

Proof. Choose any line ℓ and points P and Q on ℓ. Let R be the point on ℓ on the other side of Q from P such that $|PQ| = |QR|$. (Such a point R exists and is unique by Axiom IV.1.) Let S be any point which is not collinear with P and Q, and hence is not on ℓ. (Such a point S must exist by Axiom I.7.) Let m be the line joining Q and S (such exists and is unique by Axiom I.1) and let T be the point on m such that $|PQ| = |QT|$ and T is on the same side of Q as S (such a point T exists and is unique by Axiom IV.1). Join P to T and R to T (using Axiom I.1) to create the angle $\angle PTR$. We claim that $\angle PTR$ is a right angle.



Note first that PQ, QR, and QT were constructed to be of the same length. Thus $\triangle PQT$ and $\triangle RQT$ are both isosceles triangles. It follows that $\angle QPT = \angle QTP$ and $\angle QRT = \angle QTR$ by Lemma 0 above. It follows that $\angle PQT = 180° - 2\angle QTP$ and $\angle RQT = 180° - 2\angle QTR$. (That the sum of the interior angles of a triangle is two right angles is Theorem 20 in Hilbert's book. Hilbert doesn't give a proof, but the usual argument for this result is easy to justify in Hilbert's system.) However, $\angle PQT$ and $\angle RQT$

sum to a straight angle, so

$$\begin{aligned} 180^\circ &= \angle PQT + \angle RQT = (180^\circ - 2\angle QTP) + (180^\circ - 2\angle QTR) \\ &= 360^\circ - 2(\angle QTP + \angle QTR)\,. \end{aligned}$$

Solving this equation for $\angle QTP + \angle QTR$ gives $\angle PTR = \angle QTP + \angle QTR = 90^\circ$, as desired. □

Corollary. If A is a point on a line ℓ, then one can construct a perpendicular to ℓ at A on either side of ℓ.

Proof. This is a direct consequence of Axiom IV.4, which essentially allows one to place a congruent copy of any angle at any location. □

Lemma 2. There is a line segment of length $\sqrt{2}$.

Proof. Recall that we are assuming that some line segment, call it AB, of length 1 exists. By the Corollary to Lemma 1 we can construct a perpendicular to the line AB at A on one side of AB (either side will do). By Axiom IV.1 there is a unique point C on the perpendicular such that $|AC| = |AB|$. Then $\triangle ABC$ is a right triangle with the right angle at A, so $|BC|^2 = |AB|^2 + |AC|^2$ by the Pythagorean Theorem. Thus $|BC| = \sqrt{|AB|^2 + |AC|^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$, as desired. □

The interested reader may wish to consider here whether we need to know that the Pythagorean Theorem works in Hilbert's system in order to use it as we did in the proof Lemma 2. (If so, why? If not, why not?) In any event, it is fairly easy to prove it in Hilbert's system – Hilbert himself says so on p. 61 of his book ...

Lemma 3. There is a line segment of length $\sqrt{3}$.

Proof. Recall that we are assuming that some line segment, call it AB, of length 1 exists. By Lemma 2 there is also a line segment, say PQ, of length $\sqrt{2}$. By the Corollary to Lemma 1 we can construct a perpendicular to the line AB at A on one side of AB (either side will do). By Axiom IV.1 there is a unique point C on the perpendicular such that $|AC| = |PQ|$. Then $\triangle ABC$ is a right triangle with the right angle at A, so $|BC|^2 = |AB|^2 + |AC|^2 = |AB|^2 + |PQ|^2$ by the Pythagorean Theorem. Thus $|BC| = \sqrt{|AB|^2 + |PQ|^2} = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{1+2} = \sqrt{3}$, as desired. □

Lemma 4. There is an angle of 60°.

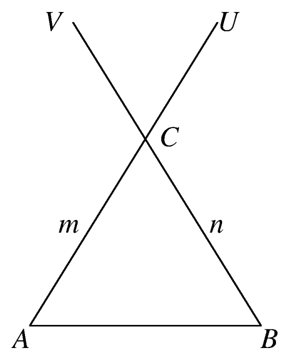
Proof. Recall that we are assuming that some line segment, call it AB, of length 1 exists. By Lemma 3 there is also a line segment, say ST, of length $\sqrt{3}$. By the Corollary to Lemma 1 we can construct a perpendicular to the line AB at A on one side of AB (either side will do). By Axiom IV.1 there is a unique point C on the perpendicular such that $|AC| = |ST|$. Then $\triangle ABC$ is a right triangle with the right angle at A, so $|BC|^2 = |AB|^2 + |AC|^2 = |AB|^2 + |ST|^2$ by the Pythagorean Theorem. It follows that $|BC| = \sqrt{|AB|^2 + |PQ|^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$.

We have thus succeeded in constructing a right triangle with sides of length 1, $\sqrt{3}$, and 2, respectively. It is well known that the angle between the sides of length 1 and 2 is 60°. □

The interested reader may wish to consider here whether we need to prove that a certain angle in a 1–$\sqrt{3}$–2 triangle measures $60°$ in Hilbert's system in order to use this fact as we did in the proof Lemma 4. (If so, why? If not, why not?)

We are – finally! – in a position to construct an equilateral triangle on a given line segment and thus prove Euclid's Proposition I-1.

Suppose we are given a line segment AB. By Lemma 4 there exists an angle of $60°$. By Axiom IV.4, we can therefore find such an angle on one side of AB at A – this really means that there is a line m passing through A with some point U on m such that $\angle UAB = 60°$. Applying Axiom IV.4 again, we can find an angle of $60°$ on the same side of AB at B – this really means that there is a line n passing through A with some point V on n such that $\angle VBA = 60°$. Since the lines m and n cannot be parallel (why not?), they must intersect in some point C.



However, any triangle with two angles of $60°$ must be equilateral. (You can check this for yourselves!) Hence $\triangle ABC$ is an equilateral triangle with base AB, as desired. ■

It's worth noting that the proof above, although not quite complete, does not appear to use the Axiom of Continuity. It does use axioms from groups I (the Axioms of Connection) and IV (the Axioms of Congruence) quite a bit, and also requires Axiom III (the Axiom of Parallels). Axiom III is needed to prove that the sum of the interior angles of a triangle is two right angles, Theorem 20 in Hilbert's *Foundations of Geometry*.