Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #7
Angle Bisectors

PRE-SOLUTION. The solutions to both problems on this assignment will exploit the following fact:

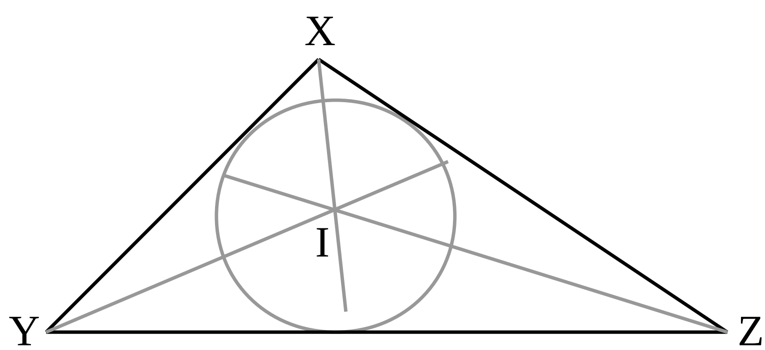
LEMMA. A point P is on the bisector of $\angle ABC$ if and only if P is equidistant from the (infinite) lines BA and BC.

PROOF. ($\Longrightarrow$) Suppose P is on the bisector of $\angle ABC$, so $\angle ABP = \angle CBP$. Let PQ be the perpendicular from P to the line AB and let PR be the perpendicular from P to the line AC. Then $\angle QBP = \angle ABP = \angle CBP = \angle RBP$, $\angle PRB = \angle PQB$ because both are right angles, and $|BP| = |BP|$, so $\triangle QBP \cong \triangle RBP$ by the ASA congruence criterion. It follows that $|PQ| = |PR|$, *i.e.* that P is equidistant from AB and AC. $\square$

($\Longleftarrow$) Suppose P is equidistant from AB and AC. Again, let PQ be the perpendicular from P to the line AB and let PR be the perpendicular from P to the line AC, so $|PQ| = |PR|$. Since $\angle PRB = \angle PQB$ because both are right angles and $|BP| = |BP|$, it follows by the SAS congruence criterion that $\triangle QBP \cong \triangle RBP$. Thus $\angle ABP = \angle QBP = \angle RBP = \angle CBP$, *i.e.* BP bisects $\angle ABC$. ■

Suppose we bisect the internal angles of $\triangle XYZ$. If we extend the lines bisecting the angles inside the triangle, they will eventually meet at a single point.

1. Show that three internal angle bisectors of a triangle are concurrent. *[5]*



The point where the three internal angle bisectors of a triangle are concurrent is called the *incentre* of the triangle.

SOLUTION. Suppose we are given $\triangle XYZ$. Let ℓ and m be the internal bisectors of $\angle YXZ$ and $\angle XYZ$, respectively, and let I be the point where ℓ and m meet. By the lemma above, I must be equidistant from XY and XZ because it is on ℓ, and it must also be equidistant from XY and YZ because it is on m. It follows that I is the same distance from XZ and YX, so the other direction of the lemma tells us that I is also on the bisector of $\angle XZY$. Thus the three internal angle bisectors of $\triangle XYZ$ are concurrent at the point I. ■

2. Show that the incentre of a triangle is the centre of a circle that touches each of the sides of the triangle at a single point. (This circle is the *incircle* of the triangle.) *[5]*

SOLUTION. Suppose I is the incentre of $\triangle XYZ$ and IU, IV, and IW are the perpendiculars from I to the sides XY, YZ, and XZ, respectively, of the triangle. Since I is equidistant from the sides of the triangle, as noted in the solution to **1** above, $|IU| = |IV| = |IW|$. Using any of IU, IV, or IW as the radius of a circle centred at I will therefore result in a circle that passes through all three of U, V, and W. Since IU, IV, and IW are radii of this circle and are perpendicular to the sides XY, YZ, and XZ, respectively, of the triangle, the sides are tangent to the circle by Proposition III-16 of Euclid's *Elements*. ■

NOTE: This gives us three centres for a triangle so far. It is traditional to denote the centroid by G, the circumcentre by C, and the incentre by I. It is worth noticing that while the centroid and the incentre must be inside the triangle, the circumcentre could be outside the triangle.

Still to come in this course are the *orthocentre*, where the three altitudes of the triangle meet, usually denoted by H, and the centre of the nine-point circle (wait for the definition :-) of the triangle, usually denoted by N. The orthocentre and the centre of the nine-point circle may also be outside the triangle.