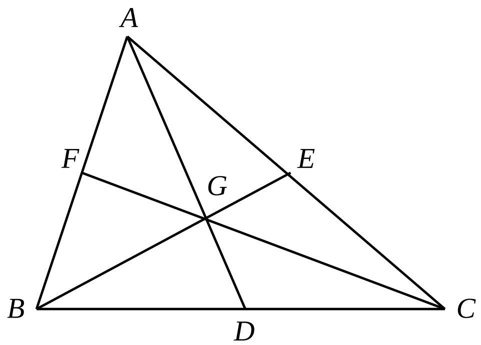
**Mathematics 2260H – Geometry I: Euclidean geometry**
TRENT UNIVERSITY, Fall 2018

**Solutions to Assignment #6**
**Medians**

NOTE. *You may work together on this assignment but you may not consult any sources other than the textbook and your notes.*

Suppose $D$, $E$, and $F$ are the midpoints of sides $BC$, $AC$, and $AB$, respectively, of $\triangle ABC$. Then $AD$, $BE$, and $CF$ are the *medians* of the triangle, and the point where the three medians intersect is the *centroid* of the triangle, usually denoted by $G$. (This is our second "centre" for a triangle after the circumcentre, which you may remember from Assignment #5.)




**1.** The medians divide up $\triangle ABC$ into six smaller triangles: $\triangle AGF$, $\triangle BGF$, $\triangle BGD$, $\triangle CGD$, $\triangle CGE$, and $\triangle AGE$. Show that the six smaller triangles have equal areas. *[4]*

SOLUTION. Since $\triangle AGF$ and $\triangle BGF$ are in the same parallels, namely $AB$ and the unique line through $G$ parallel to $AB$, and have equal bases because $F$ is the midpoint of $AB$, they have equal areas by Proposition I-38. Similar arguments show that $\triangle BGD$ and $\triangle CGD$ have equal areas, and also that $\triangle CGE$ and $\triangle AGE$ have equal areas.

Since $\triangle ABD$ and $\triangle ACD$ are in the same parallels, namely $BC$ and the unique line through $A$ parallel to $BC$, and have equal bases because $D$ is the midpoint of $BC$, they have equal areas by Proposition I-38. Since we already know that $\triangle BGD$ and $\triangle CGD$ have equal areas, it follows that $\triangle ABG$ and $\triangle ACG$ have equal areas. As each of $\triangle AGF$ and $\triangle BGF$ have half the area of $\triangle ABG$, and each of of $\triangle AGE$ and $\triangle CGE$ have half the area of $\triangle AGC$, it now follows that $\triangle AGF$, $\triangle BGF$, $\triangle AGE$, and $\triangle CGE$ all have equal areas. A similar argument starting with $\triangle ABE$ and $\triangle CBE$ shows that $\triangle AGF$, $\triangle BGF$, $\triangle BGD$, and $\triangle CGD$ all have equal areas.

Thus $\triangle AGF$, $\triangle BGF$, $\triangle BGD$, $\triangle CGD$, $\triangle CGE$, and $\triangle AGE$ have equal areas. ■

**2.** Assuming that $\triangle ABC$ is cut out of a sheet of uniform thickness and density, give an informal argument explaining why the centroid $G$ is the "balance point" of the triangle, *i.e.* the point on the triangle such that if the triangle was suspended from the point, the triangle would remain level. (Well, Archimedes, in a uniform gravity field and in the absence of other forces acting on it, and so on ... :-) *[2]*

SOLUTION. Suppose $D$, $E$, and $F$ are the midpoints of sides $BC$, $AC$, and $AB$, respectively, of $\triangle ABC$. Consider, for example, a cross-section of the triangle – a line segment! – parallel to $BC$. It is not hard to see that the middle point of this line segment, which must be its balance point in the case of uniform thickness and density, is on the median $AD$. Since the entire triangle $\triangle ABC$ can be broken up into such line segments, the median $AD$ is a balance line for $\triangle ABC$, *i.e.* the triangle will balance if placed on a straight edge along the median. Similar arguments show that the other two medians must also be balance lines of the triangle. If two balance lines cross at a point, that point must be a balance point for the entire shape [Why?], so the centroid of the triangle must be its balance point. ■

**3.** Show that the three medians of a triangle are indeed *concurrent*, that is, intersect in a single point. *[4]*

*Hints:* There are many, sometimes very different, ways to show the medians are concurrent. Here are hints for two different methods. Feel free to find other ways ...

*i.* The medians of $\triangle ABC$ are also medians of its *medial triangle*, $\triangle DEF$, which is similar to $\triangle ABC$.

*ii.* Let $G$ be the point where $BE$ and $CF$ intersect. Extend $AG$ past $G$, intersecting $BC$ at $D$ along the way, to a point $P$ such that $|AG| = |GP|$. Now show that $|BD| = |CD|$ by exploiting a certain parallelogram.

SOLUTION. We will follow the second hint. Per the hint, let $G$ be the point where $BE$ and $CF$ intersect. Extend $AG$ past $G$, intersecting $BC$ at $D$ along the way, to a point $P$ such that $|AG| = |GP|$. Connect $B$ to $P$ and $C$ to $P$.

We will first show that the quadrilateral $GBPC$ is a parallelogram. $F$ is the midpoint of side $AB$ and $G$ is the midpoint of side $AP$ in $\triangle ABP$, so $FG$is parallel to $BP$. Since $F$, $G$, and $C$ are collinear, it follows that $GC$ is also parallel to $BP$. Similarly, $E$ is the midpoint of side $AC$ and $G$ is the midpoint of side $AP$ of $\triangle ACP$, so $EG$ is parallel to $CP$. Since $E$, $G$, and $B$ are collinear, it follows that $GB$ is also parallel to $CP$. Since opposite sides of quadrilateral $GBPC$ are parallel to each other, it is a parallelogram by definition.

The diagonals of parallelogram $GBPC$ are $BC$ and $GP$, which meet at $D$. Since the diagonals of a parallelogram bidect each other, $D$ must be the midpoint of $BC$. It follows that $AD$ is a median of $\triangle ABC$, but then the three medians of the triangle are concurrent at $G$.

Note that since $|AG| = |GP|$ and $D$ is the midpoint of $GP$ (the diagonals of a parallelogram bisect each other again), it follows that $|AG| = 2|GD|$. It is not hard to adapt the reasoning used above to see that we also have $|BG| = 2|GE|$ and $|CG| = 2|GF|$. ■