Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #5
Perpendicular Bisectors

In what follows, you may use Postulates I–V (and V'), as well as Postulates A and S, and Propositions I-1 through I-30.

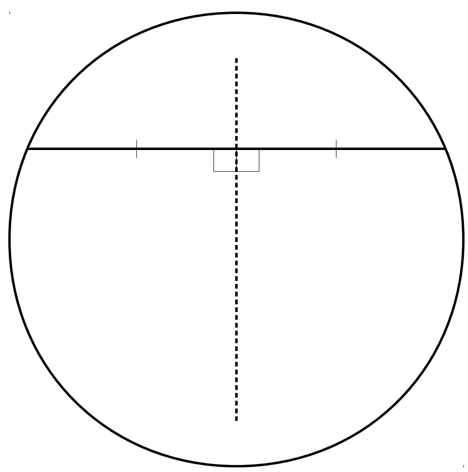
PRE-SOLUTION. It's easier to solve the given problems with the aid of the following lemma, which combines bits and pieces of Propositions I-10 through I-12 and their proofs:

LEMMA. *A point P is on the perpendicular bisector of line segment AB if and only if P is equidistant from A and B, i.e.* $|AP| = |BP|$.

PROOF. Denote the midpoint of AB by C and suppose P is a point on the perpendicular bisector of AB (which meets AB at C). If $P = C$, then it must be equidistant from A and B. Suppose P is a point on the perpendicular bisector of AB. Then $|PC| = |PC|$, $\angle PCA = \angle PCB$ since both are right angles because PC is the perpendicular bisector of AB, and $|CA| = |CB|$ because C is the midpoint of AB. It follows by the SAS congruence criterion (Proposition I-4) that $\triangle PCA \cong \triangle PCB$, and hence $|PA| = |PB|$, *i.e.* P is equidistant from A and B.

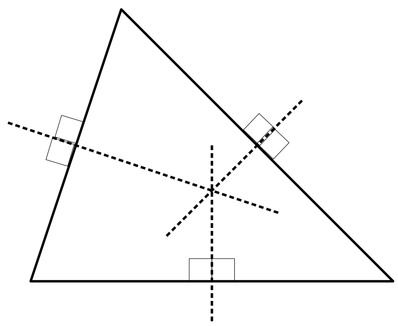
Now suppose that P is equidistant from A and B, *i.e.* $|PA| = |PB|$. If $P = C$ then it is on the perpendicular bisector of AB, which passes through C. Otherwise, if $P \neq C$, we have $|PC| = |PC|$ and $|AC| = |BC|$ (because C is the midpoint of AB). It then follows by the SSS congruence criterion (Proposition I-8) that $\triangle PCA \cong \triangle PCB$, so $\angle PCA = \angle PCB$. Since these angles add up to the straight angle $\angle ACB$, they are both right angles, and, since we also have that C is the midpoint of AB, PC is (part of) the perpendicular bisector of AB. Thus P is on the perpendicular bisector of AB. ■

1. Recall that a line segment joining two points on a circle is called a *chord* of the circle. Show that the perpendicular bisector of a chord of a circle passes through the centre of the circle. *[4]*



SOLUTION. Suppose A and B are endpoints of a chord of a circle with centre O. Then OA and OB are both radii of the circle, so $|OA| = |OB|$, *i.e.* O is equidistant from A and B. It follows by the lemma that O, the centre of the circle, is on the perpendicular bisector of the chord AB. ■

2. Show that the perpendicular bisectors of the sides of a triangle meet in a common point, which is the centre of a circle that passes through all three vertices of the triangle. *[6]*



SOLUTION. Suppose we are given $\triangle ABC$. Since AB and AC intersect at A, they are not parallel. It follows that their perpendiclar bisectors cannot be parallel either, so they must meet in a point O. Since O is on the perpendicular bisector of AB, we have $|OA| = |OB$ by the lemma, and since O is on the perpendiclar bisector of AC, we also have $|OA| = |OC$ by the lemma. Since $|OB| = |OA| = |OC|$, it now follows from the lemma that O is on the perpendicular bisector of BC. Thus the perpendicular bisectors of all three sides of $\triangle ABC$ pass through the common point O.

Draw the circle with centre O and radius OA. A obviously on this circle. Since $|OA| = |OB| = |OC|$, it follows that B and C are also on this circle, so it passes through all three vertices of the triangle. ■

NOTE. The circle passing through all three vertices of a triangle is the *circumcircle* of the triangle, and its centre, the point where the perpendicular bisectors of the sides meet, is the *circumcentre* of the triangle, usually denoted by O.