Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Fall 2018

Solutions to Assignment #4
Parallels

Recall that Euclid's Parallel Postulate is the cumbersome-seeming:

V. If a straight line falling across two other straight lines makes the sum of the internal angles on the same side less than the sum of two right angles, the two straight lines, if extended indefinitely, intersect on that side of the original straight line that the sum of the internal angles is less than the sum of two right angles.

Nowadays the favoured form of the Postulate, sometimes called Playfair's Postulate, is:

V′. Given an infinite straight line and a point not on that line, there is exactly one infinite straight line through that point that goes not intersect the given straight line.

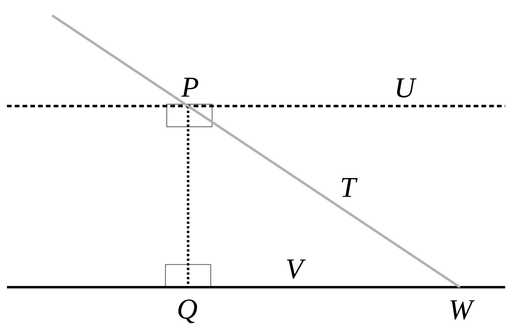
1. Show that Postulates V and V' are equivalent. You may assume Postulates I–IV, A, and S, along with Propositions I-1 through I-27 in Book I of the *Elements* (*i.e.* before Postulate V starts to be used). *[7]*

Solution. *(Postulate V $\Longrightarrow$ Postulate V')* Suppose ℓ is an infinite straight line and P is a point not on ℓ. By Proposition I-12, we can construct a line through P that is perpendicular to ℓ. Let Q be the point where this perpendicular line intersects ℓ. By Proposition I-11, we can construct a line m through P which is perpendicular to PQ (the perpendicular to ℓ we just constructed). We claim that ℓ and m are parallel.

Suppose, by way of contradiction, that ℓ and m intersect at some point R. Let S be some point on ℓ on the other side of Q from R. Then $\angle PQS = \angle QPR$ because both are right angles, since PQ is perpendicular to both ℓ and m. However, since $\angle PQS$ is an external angle and $\angle QPR$ is an opposite internal angle of $\triangle PQR$, we know from Proposition I-16 that we must have $\angle PQS > \angle QPR$, contradicting $\angle PQS = \angle QPR$. Thus m and ℓ do not intersect, and hence are parallel.

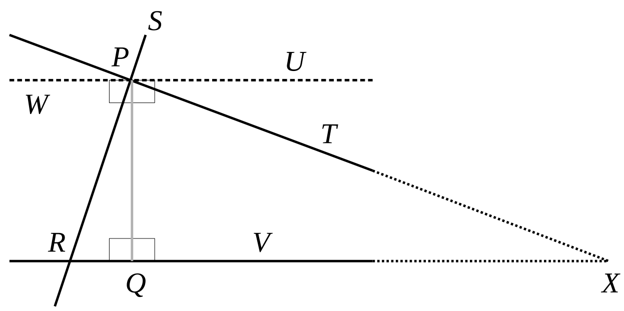
It is worth noticing that so far we have not used Postulate V, whether directly or indirectly, so far. We do need it to get the uniqueness part of Postulate V': it remains to show that m is the only line through P parallel to ℓ.

Suppose that k is a line passing through P other than m. If $k = PQ$, then it meets ℓ at Q and so is obviously not parallel to ℓ, so we may also suppose that $k \neq PQ$. Let T be a point on k between m and ℓ, U be a point on m on the same side of PQ as T, and V be a point on ℓ on the same side of PQ as T.



Then $\angle UPQ$ and $\angle VQP$ are both right angles because PQ is perpendicular to both m and ℓ. Since $\angle UPQ = \angle UPT + \angle TPQ$, we have that $\angle TPQ < \angle UPQ$, and so $\angle VQP + \angle TPQ < \angle VQP + \angle UPQ$. Since $\angle VQP$ and $\angle TPQ$ are the interior angles formed on one side of PQ by PQ falling across k and ℓ, and their sum is less than two right angles, it follows by Postulate V that k and ℓ meet at some point W on that side of PQ. Thus k and ℓ are not parallel. Hence m is the only line through P parallel to ℓ. □

(Postulate V' $\Longrightarrow$ Postulate V) Suppose PR falls across PT and RV, where T and V are on the same side of PR, and $\angle RPT + \angle PRV$ is less than two right angles. Note that this means that at least one of $\angle RPT$ and $\angle PRV$ is less than a right angle (*i.e.* acute); without loss of generality, we may suppose that $\angle PRV$ is acute. We need to show that PT and RV, extended as necessary beyond T and V, eventually intersect.



Draw the perpendicular to RV from P, meeting RV at Q. Since $\angle PRV$ is acute, Q must be on the same side of R as V is. By Postulate V' there is a line through P parallel to RV, and it's the only such line. Let U be a point on this line on the same side of PR as T and V are, and let W be a point on this line on the opposite side of PR.

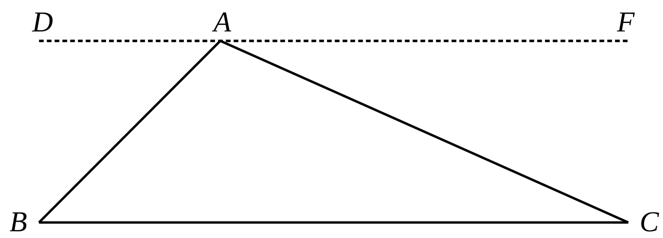
Observe that we must have $\angle WPR = \angle PRV$. If we had constructed a line through P which made these alternate angles equal, then it would be parallel to RV by Proposition I-27, and Postulate V' tells us there is only one line through P parallel to RV.

Now, since $\angle WPR + \angle RPT = \angle PRV + \angle RPT$ is less than the sum of two right angles, whereas $\angle WPR + \angle RPU = \angle WPU$ is equal to the sum of two right angles, it follows that PU and PT are (parts of) different lines, and that T is on the same side of PU as V is. Since PT is not (part of) the parallel line to RV through P, PT and RV, if extended indefinitely must meet in some point X. Since all the points on PT that are on the other side of P from T are separated from RY by the line PU, X must be on the same side of P as T, that is one the side of PR as the two interior angles that add up to less than the sum of two right angles.

Thus Postulate V holds if Postulate V' does. (Whew!) ■

2. Assuming Postulate V (along with the others :-), show that the sum of the interior angles of a triangle is a straight angle. *[3]*

SOLUTION. By **1** above, Postulate V is equivalent to Postulate V', which is what we will actually use directly. Suppose we are given any $\triangle ABC$. By Postulate V' there is an unique line through A parallel to BC. Let D be a point on the parallel line on the same side of AC as B is, and let F be a point on the parallel line on the other side of A from D, as in the diagram below.



By Proposition I-29 (*i.e.* the relevant direction of the Z-theorem), $\angle DAB = \angle ABC$ and $\angle CAF = \angle ACB$. Then the sum of the interior angles of $\triangle ABC$ is

$$\angle ABC + \angle BAC + \angle ACB = \angle DAB + \angle BAC + \angle CAF = \angle DAF\,,$$

which last is a straight angle. ∎