Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Fall 2018

Solutions to Assignment #3
Congruence Criteria

We have proved the side-angle-side (SAS) congruence criterion in class (Euclid's Proposition I-4), as well as the side-side-side (SSS) congruence criterion (Euclid's Proposition I-8).

1. Prove the angle-side-angle (ASA) congruence criterion: if $\angle ABC = \angle DEF$, $|BC| = |EF|$, and $\angle ACB = \angle DFE$, then $\triangle ABC \cong \triangle DEF$. (You may use our extended system of Postulates and Propositions I-1 through I-15.) *[5]*

Solution. Suppose $\angle ABC = \angle DEF$, $|BC| = |EF|$, and $\angle ACB = \angle DFE$. By Postulate A, we can apply $\triangle ABC$ to $\triangle DEF$ so that B is on E, BC lies along EF, and A is on the same side of EF as D. Note that C must fall on F because B is on E, BC lies along EF, and $|BC| = |EF|$.

Since $\angle ABC = \angle DEF$, B is on E, and BC lies along EF, it follows that AB lies along DE. Similarly, since $\angle ACB = \angle DFE$, C is on F, and BC lies along EF, it follows that AC lies along DF. Since AB lies along DE and AC lies along DF, AB and AC must intersect in the same point, namely A, that DE and DF do, namely D. Thus A is on D.

Since A is on D, B is on E, and C is on F, $\triangle ABC \cong \triangle DEF$. ■

It turns out that the angle-angle-side (AAS) congruence criterion also works, though we'll save that for another day. In the meantime:

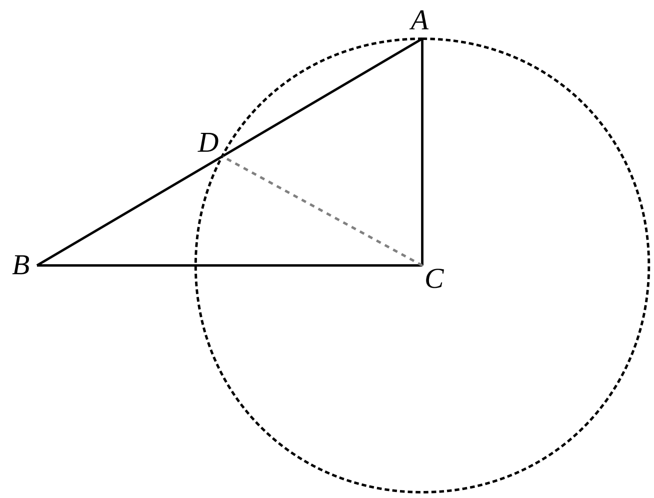
2. Does the the angle-side-side (ASS) congruence criterion work? That is, is it true that if $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $|CA| = |FD|$, then we must have $\triangle ABC \cong \triangle DEF$? Prove it that it works, using whatever method you like, or else give a counterexample and determine under what conditions, if any, the criterion does work. *[5]*

Note. The stated hypotheses are actually for AAS ... For ASS, replace $\angle BCA = \angle EFD$ by $|AB| = |DE|$. The AAS criterion is true, but ASS does not always work. Solutions for each of AAS and ASS are given below.

Solution. *(AAS)* Assume $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $|CA| = |FD|$. One of the consequences of Postulate V is that the sum of the interior angles of a triangle is equal to a straight angle. [See question **2** on Assignment #4.] Then if $\angle ABC = \angle DEF$ and $\angle BCA = \angle EFD$, we must also have $\angle BAC = \angle EDF$. Since we have $\angle ABC = \angle DEF$, $|BC| = |EF|$, and $\angle ACB = \angle DFE$, $\triangle ABC \cong \triangle DEF$ by the ASA congruence criterion. (See **1** above.) Thus the AAS congruence criterion holds. [And without Postulate V?] ■

Solution. *(ASS)* By contrast, the ASS congruence criterion does not always hold.

For a counterexample, consider $\triangle ABC$ such that $\angle ABC = 30°$, $\angle BCA = 90°$, $\angle CAB = 60°$, $|AB| = 2$, $|BC| = \sqrt{3}$, and $|CA| = 1$. Let D be the point on AB between A and B such that $|AD| = |AC| = 1$, as in the diagram below, and consider $\triangle DBC$ and $\triangle ABC$. $\angle DBC = \angle ABC$ because they are the same angle, $|BC| = |BC|$, and $|AC| = 1 = |DC|$ by our choice of D. Although we satisfy the hypotheses for the ASS criterion, it is clear that $\triangle DBC \ncong \triangle ABC$ because $\triangle DBC$ is a part, but not the whole, of $\triangle ABC$.



In spite of this example, the ASS congruence criterion does work *some* of the time. To see how this might work, consider the Law of Sines for $\triangle ABC$:

$$\frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle CAB)}{|BC|} = \frac{\sin(\angle BCA)}{|BA|}$$

Now suppose that $\angle ABC$ is at least a right angle, and that we know $|AC|$ and $|BC|$. Rearranging part of the Law of Sines gives us $\sin(\angle CAB) = \dfrac{|BC|}{|AC|}\sin(\angle ABC)$. Since the sum of the interior angles of a triangle is equal to two right angles, the other two interior angles, including $\angle CAB$, must be acute. Since $\sin(x)$ is 1–1 for $0^\circ \leq x \leq 90^\circ$, this means that we can determine $\angle CAB$, and thus also $\angle BCA$ because the sum of the three angles must be 180°. With two sides and all three angles determined, we can use the Law of Sines again to determine $|BA|$. It follows that if $\triangle ABC$ and $\triangle DEF$ satisfy the ASS hypotheses with the angle being a right or obtuse angle, then the two triangles must have all the corresponding angles and side lengths be equal, and hence be congruent.

A different, and slightly more general way, to look at it is that the ASS congruence criterion works as long as the the second side, the one opposite the given angle, is at least as long as the first side. Suppose, in the counterexample above, that we had been given instead $\angle CAB$, $|AC|$, and $|CB|$. Note that It is not hard to see that the method used in the original counterexample to construct a new triangle with the same angle and side lengths will not work because $|CB| > |AC|$. On the other hand, if we had been given $\angle CAB$, $|AB|$, and $|CB|$, the method would work because $|AB| > |CB|$. This observation allows us to use ASS in some cases where the angle is acute. Note that since the side opposite a right or obtuse angle in a triangle must be the longest side of the triangle [Why?], saying that ASS works when the opposite side is at least as long as the adjacent side also captures the case where the angle is at least a right angle. ■

Acknowledgement. The more general way of looking at when ASS works, which I don't think I had seen before, is something I got from the solution given by a student in this course, Rachel Forbes. Thanks, Rachel!