**Mathematics 2260H – Geometry I: Euclidean geometry**
TRENT UNIVERSITY, Fall 2018

**Solutions to Assignment #2**
**Congruence and Similarity**

DEFINITION. $\triangle ABC$ and $\triangle DEF$ are *congruent*, written as $\triangle ABC \cong \triangle DEF$, if all the corresponding sides and angles are equal. That is, $\triangle ABC \cong \triangle DEF$ exactly when $|AB| = |DE|$, $|AC| = |DF|$, $|BC| = |EF|$, $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. [We mentioned this in class just before doing Proposition I-4.]

Informally, this means that you can place $\triangle ABC$ over $\triangle DEF$ (possibly needing to flip it over) so that $A$ is exactly on $D$, $B$ is exactly on $E$, and $C$ is exactly on $F$; *i.e.* so that $\triangle ABC$ exactly covers $\triangle DEF$. We can make this idea a little more precise:

DEFINITION. The *rigid motions* in the plane include the *translations*, which slide all points in the plane a fixed distance in the same direction, the *rotations*, which revolve all points in the plane by some fixed angle around some fixed point, and the *reflections*, which swap all points with their mirror images across some fixed line, along with all compositions of finitely many of these three types of transformations.

**1.** Suppose $\triangle ABC \cong \triangle DEF$. Explain, informally, but as completely as you can, why one can move $\triangle ABC$ so that it exactly covers $\triangle DEF$ by no more than one translation, followed by no more than one rotation, followed by no more than one reflection. *[2]*

SOLUTION. Suppose $\triangle ABC \cong \triangle DEF$. If necessary, slide $\triangle ABC$ so that $A$ is on $D$, and then, if necessary, rotate $\triangle ABC$ about the point $A = D$ until $AB$ is on $DE$. There are now two cases:

*1.* $C$ is on the same side of $AB = DE$ as $F$.
*2.* $C$ is on the opposite side of $AB = DE$ from $F$.

In case *1,* $C$ must be on $F$, so no reflection is required; in case *2,* reflecting $\triangle ABC$ in the line $AB = DE$ leaves $AB$ on $DE$ and puts $C$ on $F$. Either way, we have $A$ on $D$, $B$ on $E$, and $C$ on $F$, so $\triangle ABC$ exactly covers $\triangle DEF$.

Why does this work? Informally, it's not too hard to convince yourself by cutting two congruent (preferably not isosceles) triangles out of paper and trying out the procedure above. Somewhat more formally, it's easy to see that one can always execute the first two steps, the slide (*i.e.* translation) and rotation, and that one or the other of the two cases must occur after the first two steps. One has to put in a little work to check that in case *1* we already have $C$ on $F$, and in case *2* that reflecting $\triangle ABC$ in the line $AB = DE$ puts $C$ on $F$.

Note that the key to the whole process, and verifying that it works, is that the definition of congruence is sensitive to the order that vertices are listed in a triangle. For example, $\triangle ABC \ncong \triangle BAC$ unless $|AC| = |BC|$. ■

Similarly – *cough, cough* – to the definition of congruence we have the following:

DEFINITION. $\triangle ABC$ and $\triangle DEF$ are *similar*, written as $\triangle ABC \sim \triangle DEF$, if all the corresponding angles are the same. That is, $\triangle ABC \sim \triangle DEF$ exactly when $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$.

That is, similar triangles have the same shape, but not necessarily the same size. Note that congruence implies similarity for triangles in the Euclidean plane, but not the other way around. We will mainly be concerned with these ideas for triangles, but the definitions can obviously be extended to other two-dimensional shapes.

**2.** Show that if $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, then $\triangle ABC \sim \triangle GHI$. *[1]*

SOLUTION. Since $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, we have, by the definition of similarity in each case, that corresponding angles are equal. This means that $\angle ABC = \angle DEF = \angle GHI$, $\angle BCA = \angle EFD = \angle HIG$, and $\angle CAB = \angle FDE = \angle IGH$, so, by the definition of similarity again, $\triangle ABC \sim \triangle GHI$. ■

Since we haven't yet developed all the Euclidean tools needed, you may, if you wish, use trigonometry and the fact that the interior angles of a triangle sum to two right angles (or one straight angle, or $\pi$ *rad*, or 180°, or ... :-) to help do the following problems.

**3.** Prove the Side-Angle-Side (SAS) similarity criterion for triangles: if $\angle ABC = \angle DEF$ and $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

SOLUTION. We will make use of the Law of Sines in each triangle:

$$\frac{\sin(\angle ABC)}{|AC|} = \frac{\sin(\angle BCA)}{|AB|} = \frac{\sin(\angle CAB)}{|BC|} \quad \text{and}$$
$$\frac{\sin(\angle DEF)}{|DF|} = \frac{\sin(\angle EFD)}{|DE|} = \frac{\sin(\angle FDE)}{|EF|}$$

Note that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$ implies that $\frac{|BC|}{|AB|} = \frac{|DF|}{|DE|}$. From this, $\angle ABC = \angle DEF$, and the Law of Sines formulas above it now follows that:

$$\sin(\angle CAB) = \sin(\angle ABC) \cdot \frac{|BC|}{|AB|} = \sin(\angle DEF) \cdot \frac{|DF|}{|DE|} = \sin(\angle FED)$$

This, in turn, implies that $\angle CAB = \angle FED$. Since the interior angles of a triangle add up to a straight angle, and $\triangle ABC$ and $\triangle DEF$ have two equal angles, the third angles must also be equal:

$$\angle BCA = \pi - \angle ABC - \angle CAB = \pi - \angle DEF - \angle FDE = \angle EFD$$

Since the corresponding angles of the two triangles are equal, $\triangle ABC \sim \triangle DEF$. ■

NOTE. There is a small hole in the argument above. Find it and fix it!

**4.** Suppose $P$ and $Q$ are the midpoints of sides $AB$ and $AC$ in $\triangle ABC$. Show that $\triangle ABC \sim \triangle APQ$ and $|BC| = 2|PQ|$. *[1]*

SOLUTION. $\angle BAC = \angle PAQ$, since it is the same angle. $|AB| = 2|AP|$ and $|AC| = 2|AQ|$ because $P$ and $Q$ are the midpoints of $AB$ and $AC$, respectively, so $\frac{|AB|}{|AP|} = 2 = \frac{|AC|}{|AQ|}$. It now follows by the SAS similarity criterion that $\triangle ABC \sim \triangle APQ$.

It remains to show that $|BC| = 2|PQ|$, which will do using the Law of Cosines:

$$\begin{aligned}
|BC|^2 &= |AB|^2 + |AC|^2 - 2 \cdot |AB| \cdot |AC| \cdot \cos(\angle BAC) \\
&= (2|AP|)^2 + (|AQ|)^2 - 2 \cdot 2|AP| \cdot 2|AQ| \cdot \cos(\angle PAQ) \\
&= 4|AP|^2 + 4|AQ|^2 - 4 \cdot 2 \cdot |AP| \cdot |AQ| \cdot \cos(\angle PAQ) \\
&= 4\left[|AP|^2 + |AQ|^2 - 2 \cdot |AP| \cdot |AQ| \cdot \cos(\angle PAQ)\right] \\
&= 4|PQ|^2 = (2|PQ|)^2
\end{aligned}$$

As line segments have non-negative lengths, it follows that $|BC| = 2|PQ|$, as desired. ■

**5.** Prove the Side-Side-Side (SSS) similarity criterion for triangles: if $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

Solution. Note that if $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = 1$, then $|AB| = |DE|$, $|BC| = |EF|$, and $|AC| = |DF|$. It now follows by the SSS congruence criterion (Proposition I-8 in Euclid's *Elements*), that $\triangle ABC \cong \triangle DEF$ and hence that $\triangle ABC \sim \triangle DEF$.

Otherwise, suppose $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} < 1$. (If $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} > 1$ instead, the same argument will work, with the roles of the triangles reversed.) We then have $|AB| < |DE|$ and $|BC| < |EF|$. Let $G$ be the point between $D$ and $E$ such that $|GE| = |AB|$ and let $H$ be the point between $E$ and $F$ such that $|EH| = |BC|$. Since $\angle GEH = \angle DEF$ and $\frac{|GE|}{|DE|} = \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|EH|}{|EF|}$, it follows by the SAS criterion for similarity (**3**) that $\triangle GEH \sim \triangle DEF$.

From $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$ we get that $|AB| = |DE| \cdot \frac{|AC|}{|DF|}$ and $|BC| = |EF| \cdot \frac{|EF|}{|DF|}$. We now use the the Law of Cosines:

$$\begin{aligned}
|GH|^2 &= |GE|^2 + |EH|^2 - 2 \cdot |GE| \cdot |EH| \cdot \cos(\angle GEH) \\
&= |AB|^2 + |BC|^2 - 2 \cdot |AB| \cdot |BC| \cdot \cos(\angle DEF) \\
&= |DE|^2 \cdot \frac{|AC|^2}{|DF|^2} + |EF|^2 \cdot \frac{|EF|^2}{|DF|^2} - 2 \cdot |DE| \cdot \frac{|AC|}{|DF|} \cdot |EF| \cdot \frac{|EF|}{|DF|} \cdot \cos(\angle DEF) \\
&= \frac{|AC|^2}{|DF|^2} \cdot \left[|DE|^2 + |EF|^2 - 2 \cdot |DE| \cdot |EF| \cdot \cos(\angle DEF)\right] \\
&= \frac{|AC|^2}{|DF|^2} \cdot |DF|^2 = |AC|^2
\end{aligned}$$

Thus $|GH| = |AC|$ in addition to $|GE| = |AB|$ and $|EH| = |BC|$, so it follows by the SSS congruence criterion that $\triangle GEH \cong \triangle ABC$, and so $\triangle GEH \sim \triangle ABC$.

Since $\triangle GEH$ is similar to both $\triangle ABC$ and $\triangle DEF$, we have $\triangle ABC \sim \triangle DEF$, as desired. ■

**6.** Suppose $P$, $Q$, and $R$ are the midpoints of sides $AB$, $AC$, and $BC$, respectively in $\triangle ABC$. Show that $\triangle RQP \sim \triangle ABC$ and $\frac{|AB||}{|RQ|} = \frac{|AC|}{|RP|} = \frac{|BC|}{|QP|} = 2$. *[2]*

Solution. Applying the length result of **4** repeatedly, we get that $|BC| = 2|QP|$, $|AB| = 2|RQ|$, and $|AC| = 2|RP|$, so $\frac{|AB||}{|RQ|} = \frac{|AC|}{|RP|} = \frac{|BC|}{|QP|} = 2$. It now follows by the SSS similarity criterion that $\triangle RQP \sim \triangle ABC$. ■