Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2018

Solutions to Assignment #1
Not Plain Planes*

Let $F = GF(3)$ be the field with three elements. That is, F is the algebraic structure whose elements are $\{0, 1, 2\}$, and which has two operations, addition and multiplication, which are done modulo 3. To be totally explicit, these operations are given by the following tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

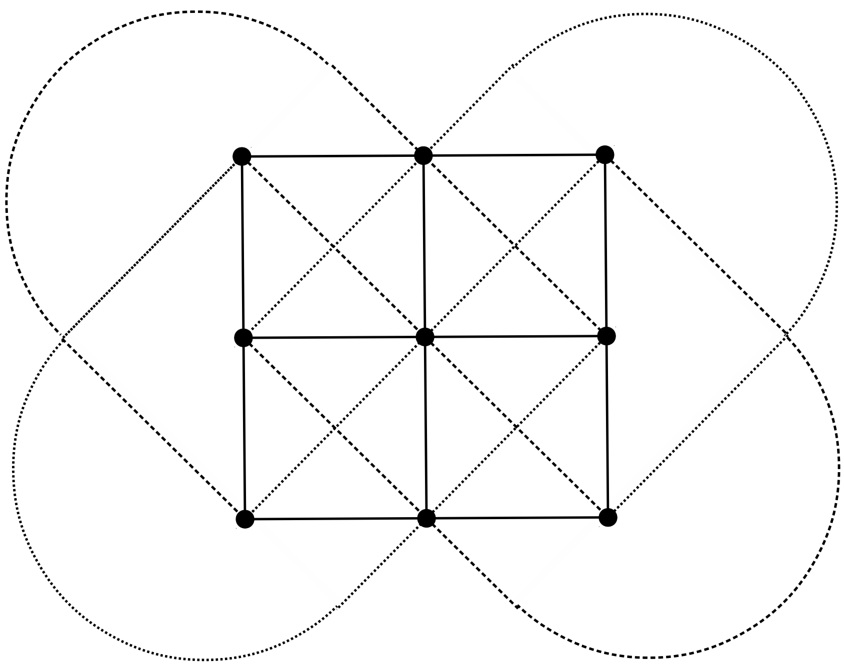
and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

It turns out that F has most of the same algebraic properties that the real numbers do, except for being seriously finite. Just as we can use the real numbers to set up coordinates (and vectors, lines, *etc.*) for familiar 2- and 3-dimensional spaces, we can use F to set up coordinates (and vectors, lines, *etc.*) for 2- and 3-dimensional finite spaces.

1. Draw a diagram of all the points and lines of F^2, the 2-dimensional Cartesian plane using F as the underlying algebraic structure. *[5]*

SOLUTION. The only points are the black circles:



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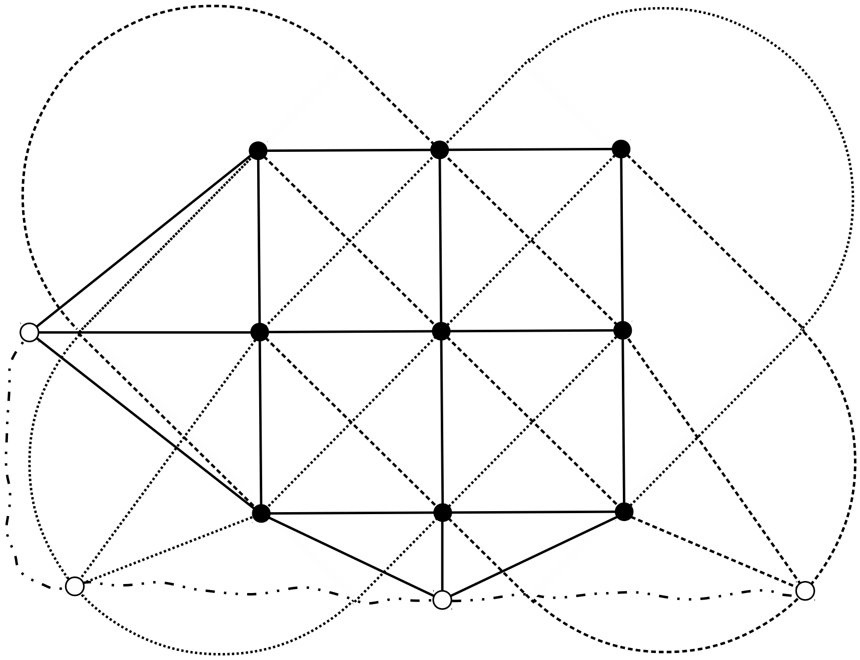
* Plain planes fly over plane plains.

2. Let F^3 be the 3-dimensional space using F as the underlying structure. $\mathcal{G} = PG(2, F)$ is the geometry defined as follows:

- The *points* of $\mathcal{G}$ are the lines through the origin (*i.e.* the 1-dimensional subspaces) of F^3.
- The *lines* of $\mathcal{G}$ are the planes through the origin (*i.e.* the 2-dimensional subspaces) of F^3.
- A given point of $\mathcal{G}$ is on a given line of $\mathcal{G}$ if and only if the corresponding line of F^3 is contained in the corresponding plane of F^3.

Draw a diagram of all the points and lines of $\mathcal{G}$. *[5]*

SOLUTION. This picture is obtained from the previous one by adding a new point for each of the four groups of three parallel lines to meet in, and then one new line connecting the four new points. The new points are the white circles.



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NOTE. Neither in **1** nor in **2** should you expect all the "lines" to be straight in your diagram ... *Nor are they in the solutions above!*

LINK. It's not too hard to do **1** by brute force in this case, but **2** is easier if you know how to embed F^2 into $\mathcal{G} = PG(2, F)$. Each point $(c, d) \in F^2$ corresponds to the subspace generated by the vector $[c, d, 1]$, each line $y = mx + b$ corresponds to the plane with normal vector $[m, -1, b]$ (note that $-1 = 2$ in F), and each line $x = a$ corresponds to the plane with normal vector $[1, 0, -a]$. The new "points" are the subspaces generated by the vectors of the form $[e, f, 0]$ (where at least one of e and f isn't 0), and the new "line" is the plane with normal vector $[0, 0, 1]$. If you want to look this stuff up look up the connection between affine and projective planes, and especially the connection between the usual (*i.e.* extended affine) coordinates and projective coordinates in projective planes.