Mathematics 2260H – Geometry I: Euclidean geometry
Trent University, Fall 2018

Assignment #2
Congruence and Similarity
Due on Friday, 21 September.

Definition. $\triangle ABC$ and $\triangle DEF$ are *congruent*, written as $\triangle ABC \cong \triangle DEF$, if all the corresponding sides and angles are equal. That is, $\triangle ABC \cong \triangle DEF$ exactly when $|AB| = |DE|$, $|AC| = |DF|$, $|BC| = |EF|$, $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$. [We mentioned this in class just before doing Proposition I-4.]

Informally, this means that you can place $\triangle ABC$ over $\triangle DEF$ (possibly needing to flip it over) so that A is exactly on D, B is exactly on E, and C is exactly on F; *i.e.* so that $\triangle ABC$ exactly covers $\triangle DEF$. We can make this idea a little more precise:

Definition. The *rigid motions* in the plane include the *translations*, which slide all points in the plane a fixed distance in the same direction, the *rotations*, which revolve all points in the plane by some fixed angle around some fixed point, and the *reflections*, which swap all points with their mirror images across some fixed line, along with all compositions of finitely many of these three types of transformations.

1. Suppose $\triangle ABC \cong \triangle DEF$. Explain, informally, but as completely as you can, why one can move $\triangle ABC$ so that it exactly covers $\triangle DEF$ by no more than one translation, followed by no more than one rotation, followed by no more than one reflection. *[2]*

Similarly – *cough, cough* – to the definition of congruence we have the following:

Definition. $\triangle ABC$ and $\triangle DEF$ are *similar*, written as $\triangle ABC \sim \triangle DEF$, if all the corresponding angles are the same. That is, $\triangle ABC \sim \triangle DEF$ exactly when $\angle ABC = \angle DEF$, $\angle BCA = \angle EFD$, and $\angle CAB = \angle FDE$.

That is, similar triangles have the same shape, but not necessarily the same size. Note that congruence implies similarity for triangles in the Euclidean plane, but not the other way around. We will mainly be concerned with these ideas for triangles, but the definitions can obviously be extended to other two-dimensional shapes.

2. Show that if $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHI$, then $\triangle ABC \sim \triangle GHI$. *[1]*

Since we haven't yet developed all the Euclidean tools needed, you may, if you wish, use trigonometry and the fact that the interior angles of a triangle sum to two right angles (or one straight angle, or π *rad*, or 180°, or … :-) to help do the following problems.

3. Prove the Side-Angle-Side (SAS) similarity criterion for triangles: if $\angle ABC = \angle DEF$ and $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

4. Suppose P and Q are the midpoints of sides AB and AC in $\triangle ABC$. Show that $\triangle ABC \sim \triangle APQ$ and $|BC| = 2|PQ|$. *[1]*

5. Prove the Side-Side-Side (SSS) similarity criterion for triangles: if $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$, then $\triangle ABC \sim \triangle DEF$. *[2]*

6. Suppose P, Q, and R are the midpoints of sides AB, AC, and BC, respectively in $\triangle ABC$. Show that $\triangle RQP \sim \triangle ABC$ and $\frac{|AB||}{|RQ|} = \frac{|AC|}{|RP|} = \frac{|BC|}{|QP|} = 2$. *[2]*