THE FIVE GROUPS OF AXIOMS.

§ 1. THE ELEMENTS OF GEOMETRY AND THE FIVE GROUPS OF AXIOMS.

Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C,. . . ; those of the second, we will call *straight lines* and designate them by the letters a, b, c,. . . ; and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ,. . . The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*.

We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated," "between," "parallel," "congruent," "continuous," etc. The complete and exact description of these relations follows as a consequence of the *axioms of geometry*. These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition. We will name these groups as follows:

I,	1–7.	Axioms of *connection*.
II,	1–5.	Axioms of *order*.
III.		Axiom of *parallels* (Euclid's axiom).
IV,	1–6.	Axioms of *congruence*.
V.		Axiom of *continuity* (Archimedes's axiom).

§ 2. GROUP I: AXIOMS OF CONNECTION.

The axioms of this group establish a connection between the concepts indicated above; namely, points, straight lines, and planes. These axioms are as follows:

I, 1. *Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.*

Instead of "determine," we may also employ other forms of expression; for example, we may say A "lies upon" a, A "is a point of" a, a "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b "have the point A in common," etc.

I, 2. *Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.*

I, 3. *Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = a$.*

We employ also the expressions: A, B, C, "lie in" a; A, B, C "are points of" a, etc.

I, 4. *Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.*

I, 5. *If two points A, B of a straight line a lie in a plane α, then every point of a lies in a.*

In this case we say: "The straight line a lies in the plane α," etc.

I, 6. *If two planes α, β have a point A in common, then they have at least a second point B in common.*

I, 7. *Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.*

Axioms I, 1–2 contain statements concerning points and straight lines only; that is, concerning the elements of plane geometry. We will call them, therefore, the *plane axioms of group I*, in order to distinguish them from the axioms I, 3–7, which we will designate briefly as the *space axioms* of this group.

Of the theorems which follow from the axioms I, 3–7, we shall mention only the following:

> Theorem 1. Two straight lines of a plane have either one point or no point in common; two planes have no point in common or a straight line in common; a plane and a straight line not lying in it have no point or one point in common.
>
> Theorem 2. Through a straight line and a point not lying in it, or through two distinct straight lines having a common point, one and only one plane may be made to pass.

§ 3. GROUP II: AXIOMS OF ORDER.[2]

The axioms of this group define the idea expressed by the word "between," and make possible, upon the basis of this idea, an *order of sequence* of the points upon a straight line, in a plane, and in space. The points of a straight line have a certain relation to one another which the word "between" serves to describe. The axioms of this group are as follows:

II, 1. *If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.*

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[2]These axioms were first studied in detail by M. Pasch in his *Vorlesungen über neuere Geometrie*, Leipsic, 1882. Axiom II, 5 is in particular due to him.

Fig. 1.

II, 2. *If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.*

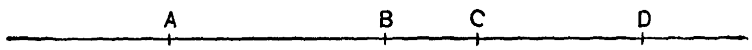

Fig. 2.

II, 3. *Of any three points situated on a straight line, there is always one and only one which lies between the other two.*

II, 4. *Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.*

Definition. We will call the system of two points A and B, lying upon a straight line, a *segment* and denote it by AB or BA. The points lying between A and B are called the *points of the segment AB* or the *points lying within the segment AB*. All other points of the straight line are referred to as the *points lying outside the segment AB*. The points A and B are called the *extremities* of the segment AB.

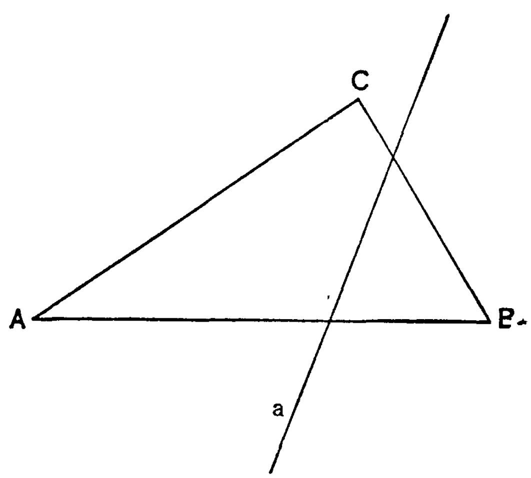

Fig. 3.

II, 5. *Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.* Axioms II, 1–4 contain statements concerning the points of a straight line only, and, hence, we will call them the *linear axioms of group II.* Axiom II, 5 relates to the elements of plane geometry and, consequently, shall be called the *plane axiom of group II.*

§ 4. CONSEQUENCES OF THE AXIOMS OF CONNECTION AND ORDER.

By the aid of the four linear axioms II, 1–4, we can easily deduce the following theorems:

THEOREM 3. Between any two points of a straight line, there always exists an unlimited number of points.

THEOREM 4. If we have given any finite number of points situated upon a straight line, we can always arrange them in a sequence A, B, C, D, $E, \ldots$, K so that B shall lie between A and C, D, $E, \ldots$, K; C between A, B and D, $E, \ldots$, K; D between A, B, C and $E, \ldots K$, etc. Aside from this order of sequence, there exists but one other possessing this property namely, the reverse order $K, \ldots$, E, D, C, B, A.

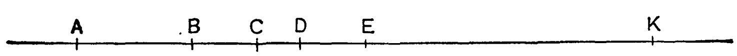

Fig. 4.

THEOREM 5. Every straight line a, which lies in a plane α, divides the remaining points of this plane into two regions having the following properties: Every point A of the one region determines with each point B of the other region a segment AB containing a point of the straight line a. On the other hand, any two points A, A' of the same region determine a segment AA' containing no point of a.

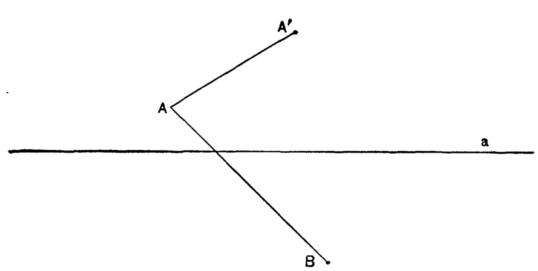

Fig. 5.

If A, A', O, B are four points of a straight line a, where O lies between A and B but not between A and A', then we may say: The points A, A' are situated *on the line a upon one and the same side of the point O,* and the points A, B are situated *on the straight line a upon different sides of the point O.*

Fig. 6.

All of the points of a which lie upon the same side of O, when taken together, are called the *half-ray* emanating from O. Hence, each point of a straight line divides it into two half-rays.

Making use of the notation of theorem 5, we say: The points A, A' lie *in the plane α upon one and the same side of the straight line a*, and the points A, B lie *in the plane α upon different sides of the straight line a.*

Definitions. A system of segments AB, BC, CD, ..., KL is called a *broken line* joining A with L and is designated, briefly, as the broken line $ABCDE \dots KL$. The points lying within the segments AB, BC, CD, ..., KL, as also the points A, B, C, D, ..., K, L, are called *the points of the broken line.* In particular, if the point A coincides with L, the broken line is called a *polygon* and is designated as the polygon $ABCD \dots K$. The segments AB, BC, CD, ..., KA are called the *sides of the polygon* and the points A, B, C, D, ..., K the *vertices.* Polygons having 3, 4, 5, ..., n vertices are called, respectively, *t*riangles, *q*uadrangles, *p*entagons, ..., n-gons. If the vertices of a polygon are all distinct and none of them lie within the segments composing the sides of the polygon, and, furthermore, if no two sides have a point in common, then the polygon is called a *simple polygon.*

With the aid of theorem 5, we may now obtain, without serious difficulty, the following theorems:

Theorem 6. Every simple polygon, whose vertices all lie in a plane α, divides the points of this plane, not belonging to the broken line constituting the sides of the polygon, into two regions, an interior and an exterior, having the following properties: If A is a point of the interior region (interior point) and B a point of the exterior region (exterior point), then any broken line joining A and B must have at least one point in common with the polygon. If, on the other hand, A, A' are two points of the interior and B, B' two points of the exterior region, then there are always broken lines to be found joining A with A' and B with B' without having a point in common with the polygon. There exist straight lines in the plane α which lie entirely outside of the given polygon, but there are none which lie entirely within it.

Theorem 7. Every plane α divides the remaining points of space into two regions having the following properties: Every point A of the one region determines with each point B of the other region a segment AH, within which lies a point of α. On the other hand, any two points A, A' lying within the same region determine a segment AA' containing no point of α.

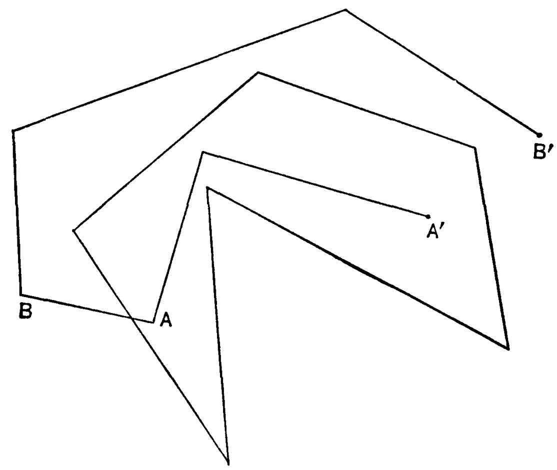

Fig. 7.

Making use of the notation of theorem 7, we may now say: The points A, A' are situated in space *upon one and the same side of the plane* α, and the points A, B are situated in space *upon different sides of the plane* α.

Theorem 7 gives us the most important facts relating to the order of sequence of the elements of space. These facts are the results, exclusively, of the axioms already considered, and, hence, no new space axioms are required in group II.

§ 5. GROUP III: AXIOM OF PARALLELS. (EUCLID'S AXIOM.)

The introduction of this axiom simplifies greatly the fundamental principles of geometry and facilitates in no small degree its development. This axiom may be expressed as follows:

> III. *In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.*

This statement of the axiom of parallels contains two assertions. The first of these is that, in the plane α, there is always a straight line passing through A which does not intersect the given line a. The second states that only one such line is possible. The latter of these statements is the essential one, and it may also be expressed as follows:

> Theorem 8. If two straight lines a, b of a plane do not meet a third straight line c of the same plane, then they do not meet each other.

For, if a, b had a point A in common, there would then exist in the same plane with c two straight lines a and b each passing through the point A and not meeting the straight line c. This condition of affairs is, however, contradictory to the second assertion contained in the axiom of parallels as originally stated. Conversely, the second part of the axiom of parallels, in its original form, follows as a consequence of theorem 8.

The axiom of parallels is a *plane axiom.*

§ 6. GROUP IV. AXIOMS OF CONGRUENCE.

The axioms of this group define the idea of congruence or displacement.

Segments stand in a certain relation to one another which is described by the word "*congruent.*"

> IV, I. *If A, B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment $A'B'$. We indicate this relation by writing*
>
> $$AB \equiv A'B'.$$
>
> *Every segment is congruent to itself; that is, we always have*
>
> $$AB \equiv AB.$$

We can state the above axiom briefly by saying that every segment can be *laid off* upon a given side of a given point of a given straight line in one and and only one way.

> IV, 2. *If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \equiv A'B$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.*

> IV, 3. *Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$, we have $AC \equiv A'C'$.*

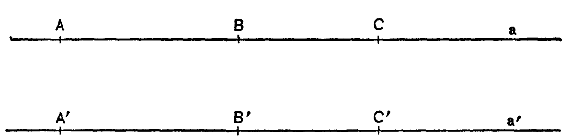

Fig. 8.

Definitions. Let α be any arbitrary plane and h, k any two distinct half-rays lying in α and emanating from the point O so as to form a part of two different straight lines. We call the system formed by these two half-rays h, k an *angle* and represent it by the symbol $\angle(h, k)$ or $\angle(k, h)$. From axioms II, 1–5, it follows readily that the half-rays h and

k, taken together with the point O, divide the remaining points of the plane a into two regions having the following property: If A is a point of one region and B a point of the other, then every broken line joining A and B either passes through O or has a point in common with one of the half-rays h, k. If, however, A, A' both lie within the same region, then it is always possible to join these two points by a broken line which neither passes through O nor has a point in common with either of the half-rays h, k. One of these two regions is distinguished from the other in that the segment joining any two points of this region lies entirely within the region. The region so characterised is called the *interior of the angle* (h, k). To distinguish the other region from this, we call it the *exterior of the angle* (h, k). The half rays h and k are called the *sides of the angle*, and the point O is called the *vertex of the angle.*

IV, 4. *Let an angle (h, k) be given in the plane α and let a straight line a' be given in a plane α'. Suppose also that, in the plane α, a definite side of the straight line a' be assigned. Denote by h' a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a'. We express this relation by means of the notation*

$$\angle(h, k) \equiv \angle(h', k')$$

Every angle is congruent to itself; that is,

$$\angle(h, k) \equiv \angle(h, k)$$

or

$$\angle(h, k) \equiv \angle(k, h)$$

We say, briefly, that every angle in a given plane can be *laid off* upon a given side of a given half-ray in one and only one way.

IV, 5. *If the angle (h, k) is congruent to the angle (h', k') and to the angle (h'', k''), then the angle (h', k') is congruent to the angle (h'', k''); that is to say, if $\angle(h, k) \equiv \angle(h', k')$ and $\angle(h, k) \equiv \angle(h'', k'')$, then $\angle(h', k') \equiv \angle(h'', k'')$.*

Suppose we have given a triangle ABC. Denote by h, k the two half-rays emanating from A and passing respectively through B and C. The angle (h, k) is then said to be the angle included by the sides AB and AC, or the one opposite to the side BC in the triangle ABC. It contains all of the interior points of the triangle ABC and is represented by the symbol $\angle BAC$, or by $\angle A$.

IV, 6. *If, in the two triangles ABC and $A'B'C'$ the congruences*

$$AB \equiv A'B',\ AC \equiv A'C',\ \angle BAC \equiv \angle B'A'C'$$

hold, then the congruences

$$\angle ABC \equiv \angle A'B'C' \text{ and } \angle ACB \equiv \angle A'C'B'$$

also hold.

Axioms IV, 1–3 contain statements concerning the congruence of segments of a straight line only. They may, therefore, be called the *linear* axioms of group IV. Axioms IV, 4, 5 contain statements relating to the congruence of angles. Axiom IV, 6 gives the connection between the congruence of segments and the congruence of angles. Axioms IV, 4–6 contain statements regarding the elements of plane geometry and may be called the *plane* axioms of group IV.

§ 7. CONSEQUENCES OF THE AXIOMS OF CONGRUENCE.

Suppose the segment AB is congruent to the segment $A'B'$. Since, according to axiom IV, 1, the segment AB is congruent to itself, it follows from axiom IV, 2 that $A'B'$ is congruent to AB; that is to say, if $AB \equiv A'B'$, then $A'B' \equiv AB$. We say, then, that the two segments are congruent to one another.

Let A, B, C, $D, \dots$, K, L and A', B', C', $D', \dots$, K', L' be two series of points on the straight lines a and a', respectively, so that all the corresponding segments AB and $A'B'$, AC and $A'C'$, BC and $B'C', \dots$, KL and $K'L'$ are respectively congruent, then *the two series of points are said to be congruent to one another*. A and A', B and $B', \dots$, L and L' are called *corresponding points* of the two congruent series of points.

From the linear axioms IV, 1–3, we can easily deduce the following theorems:

> Theorem 9. If the first of two congruent series of points A, B, C, $D, \dots$, K, L and A', B', C', $D', \dots$, K', L' is so arranged that B lies between A and C, $D, \dots$, K, L, and C between A, B and $D, \dots$, K, L, etc., then the points A', B', C', $D', \dots$, K', L' of the second series are arranged in a similar way; that is to say, B' lies between A' and C', $D', \dots$, K', L', and C' lies between A', B' and $D', \dots$, K', L', etc.

Let the angle (h, k) be congruent to the angle (h', k'). Since, according to axiom IV, 4, the angle (h, k) is congruent to itself, it follows from axiom IV, 5 that the angle (h', k') is congruent to the angle (h, k). We say, then, that the angles (h, k) and (h', k') are *congruent to one another*.

Definitions. Two angles having the same vertex and one side in common, while the sides not common form a straight line, are called *supplementary angles*. Two angles having a common vertex and whose sides form straight lines are called *vertical angles*. An angle which is congruent to its supplementary angle is called a *right angle*.

Two triangles ABC and $A'B'C'$ are said to be *congruent* to one another when all of the following congruences are fulfilled:

$$\begin{gathered} AB \equiv A'B', \quad AC \equiv A'C', \quad BC \equiv B'C', \\ \angle A \equiv \angle A', \quad \angle B \equiv \angle B', \quad \angle C \equiv \angle C'. \end{gathered}$$

> Theorem 10. (First theorem of congruence for triangles). If, for the two triangles ABC and $A'B'C'$, the congruences
>
> $$AB \equiv A'B',\ AC \equiv A'C',\ \angle A \equiv \angle A'$$
>
> hold, then the two triangles are congruent to each other.

PROOF. From axiom IV, 6, it follows that the two congruences

$$\angle B \equiv \angle B' \text{ and } \angle C \equiv \angle C'$$

are fulfilled, and it is, therefore, sufficient to show that the two sides BC and $B'C'$ are congruent. We will assume the contrary to be true, namely, that BC and $B'C'$ are not congruent, and show that this leads to a contradiction. We take upon $B'C'$ a point D' such that $BC \equiv B'D'$. The two triangles ABC and $A'B'D'$ have, then, two sides and the included angle of the one agreeing, respectively, to two sides and the included angle of the other. It follows from axiom IV, 6 that the two angles BAC and $B'A'D'$ are also congruent to each other. Consequently, by aid of axiom IV, 5, the two angles $B'A'C'$ and $B'A'D'$ must be congruent.

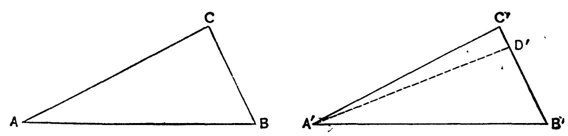

Fig. 9.

This, however, is impossible, since, by axiom IV, 4, an angle can be laid off in one and only one way on a given side of a given half-ray of a plane. From this contradiction the theorem follows.

We can also easily demonstrate the following theorem:

THEOREM 11. (Second theorem of congruence for triangles). If in any two triangles one side and the two adjacent angles are respectively congruent, the triangles are congruent.

We are now in a position to demonstrate the following important proposition.

THEOREM 12. If two angles ABC and $A'B'C'$ are congruent to each other, their supplementary angles CBD and $C'B'D'$ are also congruent.

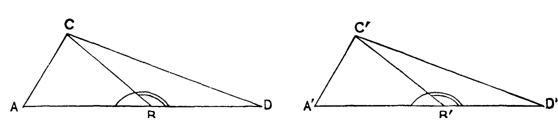

Fig. 10.

PROOF. Take the points A', C', D' upon the sides passing through B' in such a way that

$$A'B' \equiv AB, \; C'B' \equiv CB, \; D'B' \equiv DB.$$

Then, in the two triangles ABC and $A'B'C'$, the sides AB and BC are respectively congruent to $A'B'$ and $C'B'$. Moreover, since the angles included by these sides are congruent to each other by hypothesis, it follows from theorem 10 that these triangles are congruent; that is to say, we have the congruences

$$AC \equiv A'C, \ \angle BAC \equiv \angle B'A'C'.$$

On the other hand, since by axiom IV, 3 the segments AD and $A'D'$ are congruent to each other, it follows again from theorem 10 that the triangles CAD and $C'A'D'$ are congruent, and, consequently, we have the congruences:

$$CD \equiv C'D', \ \angle ADC \equiv \angle A'D'C'.$$

From these congruences and the consideration of the triangles BCD and $B'C'D'$, it follows by virtue of axiom IV, 6 that the angles CBD and $C'B'D'$ are congruent.

As an immediate consequence of theorem 12, we have a similar theorem concerning the congruence of vertical angles.

Theorem 13. Let the angle (h, k) of the plane α be congruent to the angle (h', k') of the plane α', and, furthermore, let l be a half-ray in the plane α emanating from the vertex of the angle (h, k) and lying within this angle. Then, there always exists in the plane α' a half-ray l' emanating from the vertex of the angle (h', k') and lying within this angle so that we have

$$\angle(h,l) \equiv \angle(h',l'), \ \angle(k,l) \equiv \angle(k',l').$$

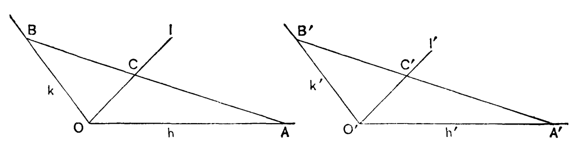

Fig. 11.

Proof. We will represent the vertices of the angles (h, k) and (h', k') by O and O', respectively, and so select upon the sides h, k, h', k' the points A, B, A', B' that the congruences

$$OA \equiv O'A', \ OB \equiv O'B'$$

are fulfilled. Because of the congruence of the triangles OAB and $O'A'B'$, we have at once

$$AB \equiv A'B', \ \angle OAB \equiv O'A'B', \ \angle OBA \equiv \angle O'B'A'.$$

Let the straight line AB intersect l in C. Take the point C' upon the segment $A'B'$ so that $A'C' \equiv AC$. Then, $O'C'$ is the required half-ray. In fact, it follows directly from these congruences, by aid of axiom IV, 3, that $BC \equiv B'C'$. Furthermore, the triangles OAC and $O'A'C'$ are congruent to each other, and the same is true also of the triangles OCB and $O'B'C'$. With this our proposition is demonstrated.

In a similar manner, we obtain the following proposition.

Theorem 14. Let h, k, l and h', k', l' be two sets of three half-rays, where those of each set emanate from the same point and lie in the same plane. Then, if the congruences

$$\angle(h,l) \equiv \angle(h',l'),\ \angle(k,l) \equiv \angle(k',l')$$

are fulfilled, the following congruence is also valid; viz.:

$$\angle(h,k) \equiv \angle(h',k').$$

By aid of theorems 12 and 13, it is possible to deduce the following simple theorem, which Euclid held–although it seems to me wrongly–to be an axiom.

Theorem 15. All right angles are congruent to one another.

Proof. Let the angle BAD be congruent to its supplementary angle CAD, and, likewise, let the angle $B'A'D'$ be congruent to its supplementary angle $C'A'D'$. Hence the angles BAD, CAD, $B'A'D'$, and $C'A'D'$ are all right angles. We will assume that the contrary of our proposition is true, namely, that the right angle $B'A'D'$ is not congruent to the right angle BAD, and will show that this assumption leads to a contradiction. We lay off the angle $B'A'D'$ upon the half-ray AB in such a manner that the side AD'' arising from this operation falls either within the angle BAD or within the angle CAD. Suppose, for example, the first of these possibilities to be true. Because of the congruence of the angles $B'A'D'$ and BAD'', it follows from theorem 12 that angle $C'A'D'$ is congruent to angle CAD'', and, as the angles $B'A'D'$ and $C'A'D'$ are congruent to each other, then, by IV, 5, the angle BAD'' must be congruent to CAD''.

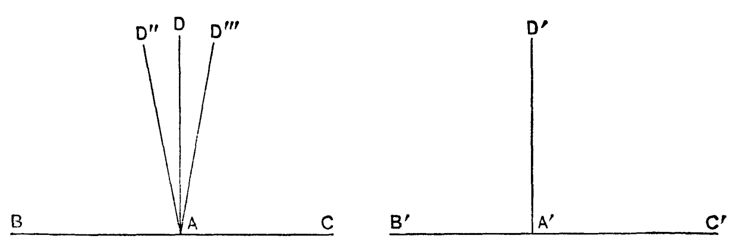

Fig. 12.

Furthermore, since the angle BAD is congruent to the angle CAD, it is possible, by theorem 13, to find within the angle CAD a half-ray AD''' emanating from A, so that the angle BAD'' will be congruent to the angle CAD''', and also the angle DAD'' will be congruent to the angle DAD'''. The angle BAD'' was shown to be congruent to the angle CAD'' and, hence, by axiom IV, 5, the angle CAD'', is congruent to the angle CAD'''. This, however, is not possible; for, according to axiom IV, 4, an angle can be laid off in a plane upon a given side of a given half-ray in only one way. With this our proposition is demonstrated. We can now introduce, in accordance with common usage, the terms "*acute angle*" and "*obtuse angle*."

The theorem relating to the congruence of the base angles A and B of an equilateral triangle ABC follows immediately by the application of axiom IV, 6 to the triangles ABC and BAC. By aid of this theorem, in addition to theorem 14, we can easily demonstrate the following proposition.

> Theorem 16. (Third theorem of congruence for triangles.) If two triangles have the three sides of one congruent respectively to the corresponding three sides of the other, the triangles are congruent.

Any finite number of points is called a *figure*. If all of the points lie in a plane, the figure is called a *plane figure*.

Two figures are said to be *congruent* if their points can be arranged in a one-to-one correspondence so that the corresponding segments and the corresponding angles of the two figures are in every case congruent to each other.

Congruent figures have, as may be seen from theorems 9 and 12, the following properties: Three points of a figure lying in a straight line are likewise in a straight line in every figure congruent to it. In congruent figures, the arrangement of the points in corresponding planes with respect to corresponding lines is always the same. The same is true of the sequence of corresponding points situated on corresponding lines.

The most general theorems relating to congruences in a plane and in space may be expressed as follows:

> Theorem 17. If $(A, B, C, \ldots)$ and $(A', B', C', \ldots)$ are congruent plane figures and P is a point in the plane of the first, then it is always possible to find a point P in the plane of the second figure so that $(A, B, C, \ldots, P)$ and $(A', B', C', \ldots, P')$ shall likewise be congruent figures. If the two figures have at least three points not lying in a straight line, then the selection of P' can be made in only one way.

> Theorem 18. If $(A, B, C, \ldots)$ and $(A', B', C', \ldots =$ are congruent figures and P represents any arbitrary point, then there can always be found a point P' so that the two figures $(A, B, C, \ldots, P)$ and $(A', B', C', \ldots, P')$ shall likewise be congruent. If the figure $(A, B, C, \ldots, P)$ contains at least four points not lying in the same plane, then the determination of P' can be made in but one way.

This theorem contains an important result; namely, that all the facts concerning space which have reference to congruence, that is to say, to displacements in space, are (by the addition of the axioms of groups I and II) exclusively the consequences of the six linear and plane axioms mentioned above. Hence, it is not necessary to assume the axiom of parallels in order to establish these facts.

If we take, in, addition to the axioms of congruence, the axiom of parallels, we can then easily establish the following propositions:

> Theorem 19. If two parallel lines are cut by a third straight line, the alternate-interior angles and also the exterior-interior angles are congruent Conversely, if the alternate-interior or the exterior-interior angles are congruent, the given lines are parallel.

Theorem 20. The sum of the angles of a triangle is two right angles.

Definitions. If M is an arbitrary point in the plane a, the totality of all points A, for which the segments MA are congruent to one another, is called *a circle*. M is called the *centre of the circle*.

From this definition can be easily deduced, with the help of the axioms of groups III and IV, the known properties of the circle; in particular, the possibility of constructing a circle through any three points not lying in a straight line, as also the congruence of all angles inscribed in the same segment of a circle, and the theorem relating to the angles of an inscribed quadrilateral.

§ 8. GROUP V. AXIOM OF CONTINUITY. (ARCHIMEDEAN AXIOM.)

This axiom makes possible the introduction into geometry of the idea of continuity. In order to state this axiom, we must first establish a convention concerning the equality of two segments. For this purpose, we can either base our idea of equality upon the axioms relating to the congruence of segments and define as "*equal*" the correspondingly congruent segments, or upon the basis of groups I and II, we may determine how, by suitable constructions (see Chap. V, § 24), a segment is to be laid off from a point of a given straight line so that a new, definite segment is obtained "*equal*" to it. In conformity with such a convention, the axiom of Archimedes may be stated as follows:

V. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points A_2, A_3, $A_4, \dots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments

$$AA_1,\ A_1A_2,\ A_2A_3,\ A_3A_4,\ \dots$$

be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.

The axiom of Archimedes is a *linear* axiom.

Remark.[3] To the preceeding five groups of axioms, we may add the following one, which, although not of a purely geometrical nature, merits particular attention from a theoretical point of view. It may be expressed in the following form:

Axiom of Completeness.[4] *(Vollständigkeit): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.*

[3] Added by Professor Hilbert in the French translation.—*Tr.*

[4] See Hilbert, "Ueber den Zahlenbegriff," *Berichte der deutschen Mathematiker-Vereinigung*, 1900.

This axiom gives us nothing directly concerning the existence of limiting points, or of the idea of convergence. Nevertheless, it enables us to demonstrate Bolzano's theorem by virtue of which, for all sets of points situated upon a straight line between two definite points of the same line, there exists necessarily a point of condensation, that is to say, a limiting point. From a theoretical point of view, the value of this axiom is that it leads indirectly to the introduction of limiting points, and, hence, renders it possible to establish a one-to-one correspondence between the points of a segment and the system of real numbers. However, in what is to follow, no use will be made of the "axiom of completeness."