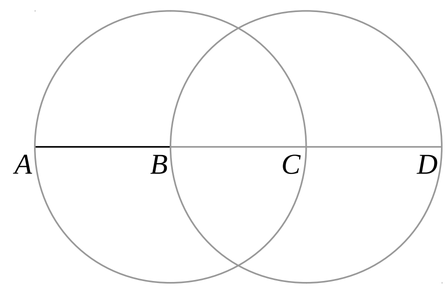
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2016

Solutions to the Quizzes

Quiz #1. Wednesday, 13 September. *[10 minutes]*

1. Suppose you are given a line (segment) AB. Using whichever you like of Postulates I–V, A, and S, as well as Propositions I-1 and I-2, show that there is a point D such that AD is three times as long as AB. *[5]*

SOLUTION. You can probably guess how this works from the picture ...



... but here is the construction of AD, step by step:

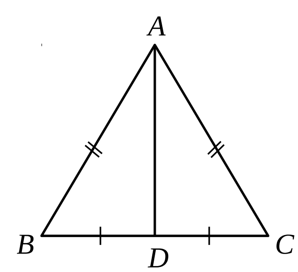
1. Draw a circle with radius AB and centre B. [Postulate III]
2. Extend AB past B until it meets the circle at C. [Postulates II and S]
3. Draw a circle with radius BC and centre C. [Postulate III]
4. Extend BC (which is part of AC) past C until it meets the second circle at D. [Postulates II and S]

This works because AD is made up of three pieces, namely AB, BC, and CD, which are equal in length. AB and BC are equal in length because both are radii of the first circle, and BC and CD are equal in length because both are radii of the second circle. (By some Common Notion or other, it follows that AB and CD are also equal in length, if anyone dares to object.) Thus AD is three times as long as AB. ■

Quiz #2. Wednesday, 21 September. *[10 minutes]*

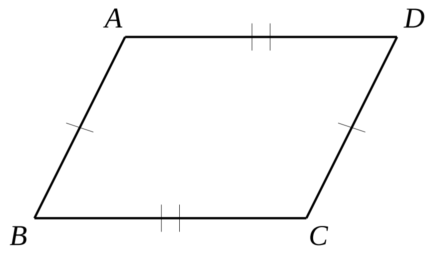
1. Suppose that $\triangle ABC$ is equilateral and that D is the midpoint of side BC. Show that $\angle BAD = \angle CAD$. *[5]*

SOLUTION. Since $\triangle ABC$ is equilateral, $AB = AC$, and it follows by Proposition I-5 that $\angle ABD = \angle ABC = \angle ACB = \angle ACD$. We are also given that $BD = CD$, as D is the midpoint of BC. Now apply $\triangle ABD$ to $\triangle ACD$ so that B goes on C, BA lies along CA, and D is on the same side as D. Since $BA = CA$, A falls on A; since $\angle ABD = \angle ACD$, BD must lie along CD, and since $BD = CD$, it also follows that D goes on D. Since B goes on C, A on A, and D on D, it follows that $\angle BAD = \angle CAD$. ■



Quiz #3. Wednesday, 28 September. *[10 minutes]*

1. Suppose $ABCD$ is a quadrilateral such that $AB = CD$ and $BC = AD$. Show that $\angle BAD = \angle BCD$. *[5]*

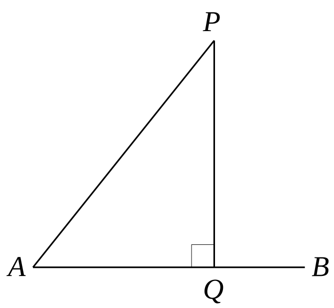


Solution. Join B to D. Then $DB = BD$ because things are equal to themselves, and we were given that $AB = DC$ and $AD = CB$. Thus $\triangle BAD \cong \triangle BCD$ by the Side-Side-Side congruence criterion (*i.e.* Proposition I-8), and it follows that $\angle BAD = \angle BCD$. ■

Quiz #4. Wednesday, ~~5~~ 12 October. *[10 minutes]*

1. Suppose AB is a line, P is point not on AB, and Q is a point on AB between A and B such that $\angle AQP$ is a right angle. Show that $PA > PQ$. *[5]*

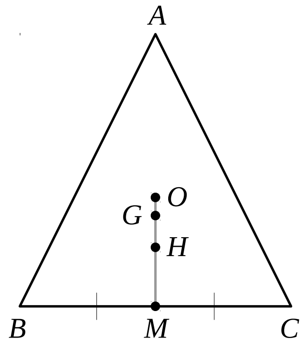
Solution. Here's the picture:



Note first that since $\angle AQP$ is a right angle and $\angle AQB = \angle AQP + \angle PQB$ is a right angle, $\angle PQB$ is also a right angle and so is equal to $\angle AQP$. Since $\angle PQB$ is an exterior angle of $\triangle APQ$, it is greater than the opposite interior angle $\angle PAQ$ of the triangle by Proposition I-16, so it follows that $\angle PQB = \angle AQP > \angle PAQ$. By Proposition I-19, the greater angle subtends the greater side in $\triangle APQ$, so $PA > PQ$, as required. ■

Quiz #6. Wednesday, 9 November. *[10 minutes]*

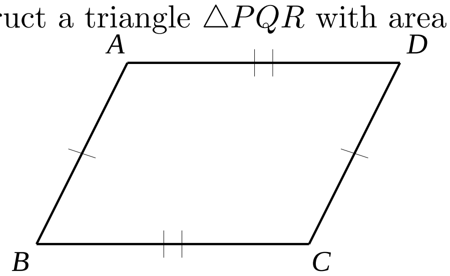
1. Recall that the circumcentre O, centroid G, and orthocentre H of $\triangle ABC$ are in a straight line, which is called the Euler line of the triangle. Suppose that the midpoint M of side BC is also on this line. Show that $\triangle ABC$ must be isosceles. *[5]*



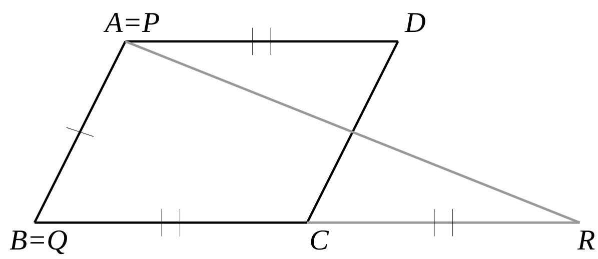
Solution. Since the median from vertex A to M passes through the centroid G, and both G and M are on the Euler line, vertex A is also on the Euler line. Thus AM also passes through the orthocenter H, so it must be the altitude from A, and so $\angle AMB$ and $\angle AMC$ are both right angles and equal to each other. Since $AM = AM$ and we are given that $BM = CM$ (since M is the midpoint of BC), it follows by the Side-Angle-Side congruence criterion that $\triangle AMB \cong \triangle AMC$. This, in turn, means that $AB = AC$, so $\triangle ABC$ is isosceles. ■

Quiz #~~5 6~~ 7. ~~Wednesday~~ Monday, ~~19 October 2 7~~ 14 November. *[10 minutes]*

1. Suppose $ABCD$ is a parallelogram, with $AB \parallel DC$ and $BC \parallel AD$ (so $AB = DC$ and $BC = AD$, too). Construct a triangle $\triangle PQR$ with area equal to $ABCD$. *[5]*



Solution. Extend BC past C to R such that $BC = CR$. Let $P = A$ and $Q = B$, and consider $\triangle PQR$.

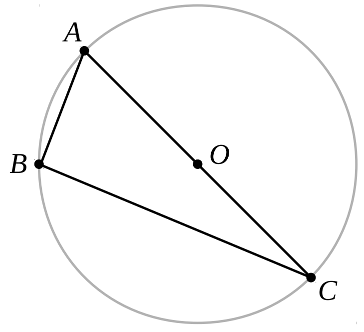


$\triangle PQR$ has the same height as the parallelogram $ABCD$ and a base that is twice as long. If we denote the common height by h and the base of the parallelogram (*i.e.* the length of BC) by b, then the area of the triangle is $\frac{1}{2}(2b)h = bh$, which is the area of the parallelogram, as required. ■

Quiz #8. Wednesday, 16 November. *[10 minutes]*

1. Suppose that the circumcentre O is on the side AC of $\triangle ABC$. Where does the orthocentre H of $\triangle ABC$ have to be? *[5]*

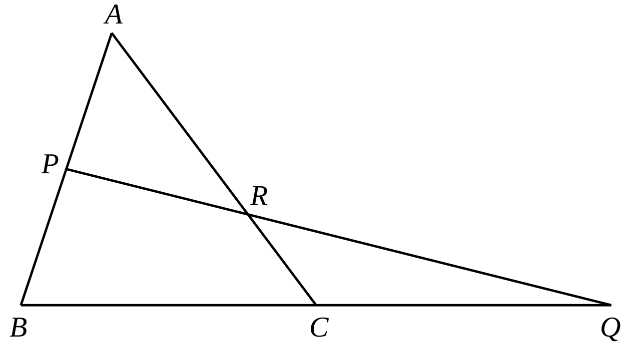
Solution. The circumcircle of $\triangle ABC$, which has centre O, passes through all three vertices of the triangle.



Since AC passes through O, AC is a diameter of the circumcircle, and, since we also have that B is on the circumcircle, $\angle ABC$ is a right angle by Thales' Theorem. This means that the altitude from A is AB and that the altitude from C is BC. Since B is the point where these altitudes meet, B is the orthocentre of $\triangle ABC$, *i.e.* $H = B$. ■

Quiz #10. Wednesday, 23 November. *[10 minutes]*

1. Suppose that we are given $\triangle ABC$, and collinear points P, Q, and R such that P is on AB between A and B, Q is on BC beyond C, and R is on AC between A and C, as in the diagram below.

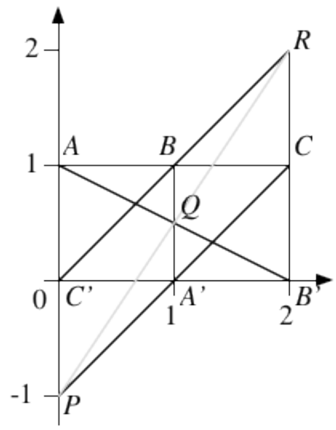


Show that $\frac{AB}{BP} \cdot \frac{PQ}{QR} \cdot \frac{RC}{CA} = -1$. *[5]*

Solution. Consider instead $\triangle APR$. B is on AP beyond P, Q is on PR beyond R, and C is on AR beyond R, with B, C, and Q collinear. It follows by Menelaus' Theorem that $\frac{AB}{BP} \cdot \frac{PQ}{QR} \cdot \frac{RC}{CA} = -1$. ■

Quiz #11. Wednesday, 30 November. *[10 minutes]*

1. Suppose $A = (0, 1)$, $B = (1, 1)$, $C = (1, 2)$, $A' = (1, 0)$, $B' = (2, 0)$, and $C' = (0, 0)$ in the Cartesian plane. Let $P = AC' \cap A'C$, $Q = AB' \cap A'B$, and $R = BC' \cap B'C$.



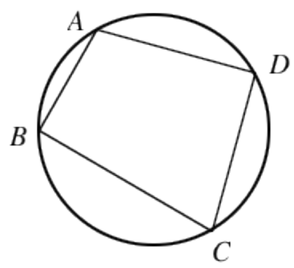
Verify that Pappus' Theorem holds in this instance by showing directly that P, Q, and R are collinear. *[5]*

Solution. It's easy to see that AC' is the line $x = 0$ and $A'C$ is the line $y = x - 1$, so their intersection is the point $P = (0, -1)$. Similarly, since AB' is the line $y = -\frac{1}{2}x + 1$ and $A'B$ is the line $x = 1$, their intersection is the point $Q = \left(1, \frac{1}{2}\right)$, and since BC' is the line $y = x$ and $B'C$ is the line $x = 2$, their intersection is the point $R = (2, 2)$. [Alternatively, one could just read off the slopes and intercepts of the lines and the coordinates of the points from the sketch …]

Since $P = (0, -1)$, $Q = \left(1, \frac{1}{2}\right)$, and $R = (2, 2)$ are all on the straight line $y = \frac{3}{2}x - 1$, they are collinear, as desired. ■

Quiz #12. Wednesday, 7 December. *[10 minutes]*

1. Suppose $ABCD$ is a convex quadrilateral inscribed in a circle.



Show that $\angle ABC + \angle CDA = 2$ right angles. *[5]*

SOLUTION. Let O be the centre of the circle. Then $\angle ABC = \frac{1}{2}\angle AOC$ and $\angle CDA = \frac{1}{2}\angle COA$. It follows that

$$\begin{aligned}\angle ABC + \angle CDA &= \frac{1}{2}\angle AOC + \frac{1}{2}\angle COA = \frac{1}{2} \cdot 2 \cdot \text{straight angles} \\ &= 1 \text{ straight angle} = 2 \text{ right angles}\,,\end{aligned}$$

as desired. ■