Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2015

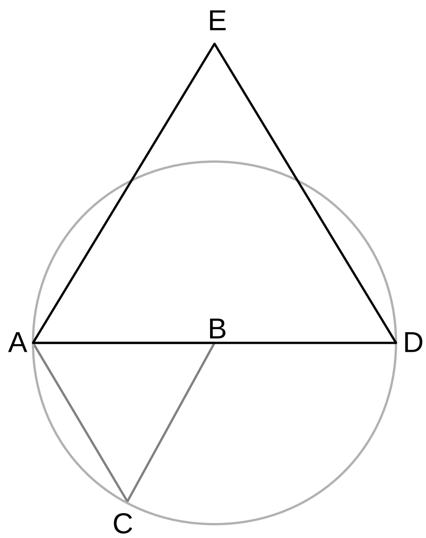
Solutions to the Quizzes

Quiz #1. *Monday, 21 September, 2015. [10 minutes]*

1. Given an equilateral triangle $\triangle ABC$, use whichever you wish of Postulates I-V, S, and A, as well as Propositions I-1 through I-4, to show that there is an equilateral triangle $\triangle DEF$ whose sides are twice as long as the sides of $\triangle ABC$. *[5]*

SOLUTION. The key is to construct a line [segment] which is twice the length of a side of $\triangle ABC$. There is more than one way to do so; here is a simple one:

Draw a circle with centre B and radius AB. (Postulate III)
Extend AB past B until it meets the circle at D. (Postulates I and S)
Construct an equilateral triangle $\triangle ADE$ with base AD. (Proposition I-1)



$\triangle ADE$ does the job: since it is equilateral, every side is just as long as AD, which is twice as long as AB, and hence also twice as lomg the other sides of equilateral triangle $\triangle ABC$. (If you really want the larger triangle to be named $\triangle DEF$, just let $F = A$.) ■

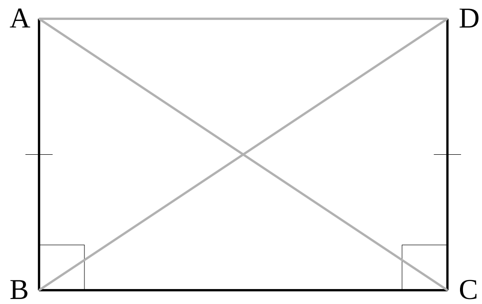
Quiz #2. *Monday, 28 September, 2015. [10 minutes]*

1. Suppose $\triangle ABC$ is isosceles with $|AB| = |AC|$, and D is a point on BC between B and C such that $\angle BAD = \angle CAD$. Show that $|BD| = |CD|$. *[5]*

SOLUTION. $|AB| = |AC|$ and $\angle BAD = \angle CAD$ are given, and $|AD| = |AD|$ because each thing is equal to itself. It follows by the Side-Angle-Side congruence criterion, also known as Proposition I-4, that $\triangle BAD \cong \triangle CAD$. From this, in turn, it follows that $|BD| = |CD|$, *i.e.* D is halfway between B and D. ■

Quiz #3. *Monday, 5 October, 2015. [15 minutes]*

1. Suppose A, B, C, and D are four points such that $|AB| = |CD|$ and $\angle ABC = \angle DCB$ are right angles. Show that $\triangle BAD \cong \triangle CDA$. *[5]*



Solution. Connect A to D, A to C, and B to D. It is given that $|AB| = |CD|$ and $\angle ABC = \angle DCB$, and obviously $|BC| = |CB|$, so $\triangle ABC \cong \triangle DCB$ by the SAS congruence criterion (*i.e.* Proposition I-4). It follows from this that $|AC| = |DB|$; since we also have $|AB| = |CD|$ (given) and $|AD| = |DA|$, the SSS congruence criterion (*i.e.* Proposition I-8) then gives that $\triangle BAD \cong \triangle CDA$, as desired. $\blacksquare$

Quiz #4. *Thursday, 15 October, 2015. [10 minutes]*

1. Show that the sum of the interior angles of any triangle is less than three right angles. *[5]*

Solution. By Proposition I-17, the sum of any two interior angles of triangles is less than two right angles, *i.e.* less than $2 \cdot \frac{\pi}{2} = \pi$ radians. If α, β and γ are the interior angles of some triangle, this means that $\alpha + \beta < \pi$, $\beta + \gamma < \pi$, and $\gamma + \alpha < \pi$. It follows that

$$2(\alpha + \beta + \gamma) = (\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) < \pi + \pi + \pi = 3\pi,$$

so $\alpha + \beta + \gamma < \frac{3}{2}\pi =$ three right angles, as desired. $\blacksquare$

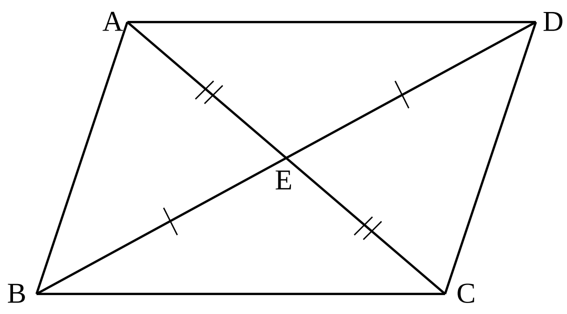
Quiz #5. *Monday, 19 October, 2015. [15 minutes]*

1. Suppose $\triangle ABC$ is isosceles, with $|AB| = |AC|$, and D is a point on BC between B and C. Show that $|AD| < |AB|$. *[5]*

Solution. $\angle ADB$ is an exterior angle of $\triangle ADC$, so it is greater than the opposite interior angle $\angle ACD$ of $\triangle ADC$. Since $\angle ACD = ACB$ and, because $\triangle ABC$ is isosceles with $|AB| = |AC|$, $\angle ABC = \angle ACB$, it follows that $\angle ADB > \angle ACD = \angle ACB = \angle ABC$. Applying Proposition I-18 to $\triangle ADB$, because $\angle ADB > \angle ABC = \angle ABD$, the side subtended by $\angle ADB$, namely AB, is longer than the side subtended by $\angle ABD$, namely AD, *i.e.* $|AB| > |AD|$, as required. $\blacksquare$

Quiz #6. *Monday, 2 November, 2015. [10 minutes]*

1. Suppose $ABCD$ is a quadrilateral, E is the point where its diagonals, AC and BD intersect, and $|AE| = |CE|$ and $|BE| = |DE|$, as in the diagram below. Show that $ABCD$ is a parallelogram, *i.e.* that $AD \parallel BC$ and $AB \parallel DC$. *[5]*



Solution. $\angle AED = \angle CEB$ because they are opposite angles and we are given that $|AE| = |CE|$ and $|BE| = |DE|$. It follows by the Side-Angle-Side congruence criterion that $\triangle AEB \cong \triangle CEB$. Thus $\angle ADE = \angle CBE$, and so the Z-theorem implies that $AD \parallel BC$.

Similarly, $\angle AEB = \angle CED$ because they are opposite angles and we are given that $|AE| = |CE|$ and $|BE| = |DE|$. It follows by the Side-Angle-Side congruence criterion that $\triangle AEB \cong \triangle CED$. Thus $\angle BAE = \angle DCE$, and so the Z-theorem implies that $AB \parallel DC$.

Since $AD \parallel BC$ and $AB \parallel DC$, $ABCD$ is a parallelogram. ∎

Quiz #7. *Monday, 9 November, 2015. [10 minutes]*

1. Suppose $ABCD$ is a parallelogram, with $AB \parallel DC$ and $AD \parallel BC$. Show that $\triangle ABC$ has the same area as $\triangle DBC$. *[5]*

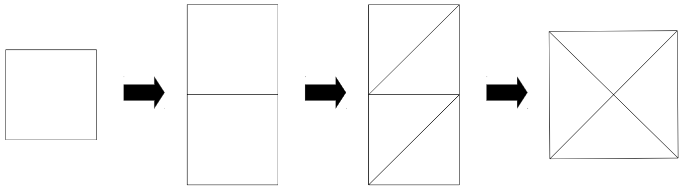
Solution. $\triangle ABC$ and $\triangle DBC$ have the common base BC. Since $AD \parallel toBC$ it is just as far from A to BC as it is from D to BC, so $\triangle ABC$ and $\triangle DBC$ have the same height. Since the area of a triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$, it follows that the two triangles have the same area. ∎

Quiz #8. *Monday, 16 November, 2015. [10 minutes]*

Do *one* (1) of the following problems:

1. Suppose $ABCD$ is a square with area 1. Use it to construct a square of area 2. *[5]*
2. Suppose a circle with centre C has radius 2, and A and B are points on the circle such that $|CD| = 1$ for the midpoint D of the chord AB. Find the area of $\triangle ABC$. *[5]*

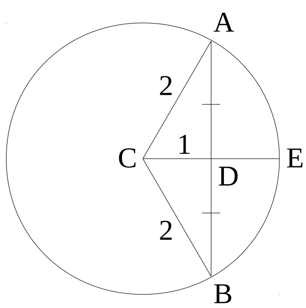
SOLUTIONS. 1. Here is a skeleton solution in visual form:



A more detailed solution follows:

Make another square of the same size as $ABCD$ by constructing a square $BEFC$ on the line segment BC, on the opposite side from AD. Draw the diagonals BD and EC and observe that the triangles $\triangle ABD$, $\triangle CDB$, $\triangle BEC$, and $\triangle FCE$ are all congruent each other by the Side-Angle-Side (SAS) congruence criterion: each has two sides of length 1 with a right angle between them. Moreover, each is an isosceles triangle, so the two interior angles that are not right are each half of a right angle (*i.e.* 45° or $\frac{\pi}{4}$ radians). Rearranging the four triangles into a quadrilateral as indicated in the last part of the diagram above gives a square because each side is the same length as each is the long side of one of the four congruent right triangles, and each interior angle is a right angles because it is made up of two angles which are each half of a right angle. $\square$

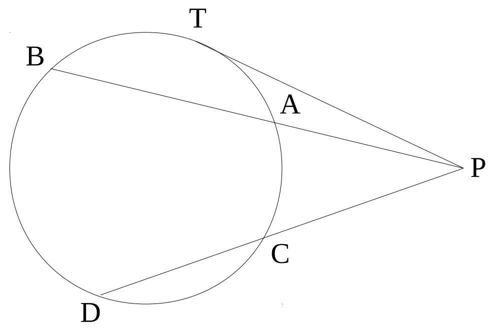
2. Extend CD past D until it meets the circle at E, so CE is a radius and has length 2.



This radius meets the chord AB in the chord's midpoint D, and so the radius and chord must be perpendicular (Proposition III.3). This means that $\triangle ACD$ is a right triangle, so the Pythagorean Theorem tells us that $|AC|^2 = |CD|^2 + |AD|^2$. Since $|AC| = 2$ and $|CD| = 1$, it follows that $|AD| = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$, and so $|AB| = |AD| + |DB| = |AD| + |AD| = 2\sqrt{3}$. Thus $\triangle ABC$ has base $|AB| = 2\sqrt{3}$ and height $|CD| = 1$ – recall that $CD \perp AB$ – so the area of the triangle is area $\frac{1}{2} \cdot 2\sqrt{3} \cdot 1 = \sqrt{3}$. ■

Quiz #9. *Monday, 23 November, 2015. [10 minutes]*

1. Show that if AB and CD are chords of a circle that, when extended past A and C respectively, meet at a point P outside the circle, then $|PA| \cdot |PB| = |PC| \cdot |PD|$. *[5]*

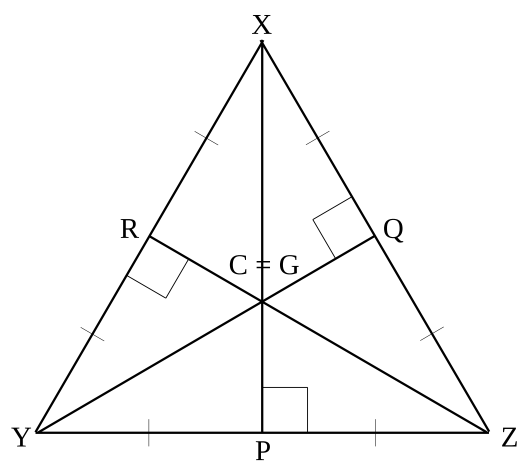


Solution. Let T be point on the circle such that PT is tangent to the circle. (There are two such points and either one will do.) By Proposition III.36 applied to PT and one each of the chords at a time, it follows that $|PA| \cdot |PB| = |PT|^2 = |PC| \cdot |PD|$. ■

Quiz #10. *Monday, 30 November, 2015. [10 minutes]*

1. Suppose the circumcentre C and the centroid G of $\triangle XYZ$ are the same point. Show that $\triangle XYZ$ is equilateral. *[5]*

Solution. Let P, Q, and R be points where the extensions of XC, YC, and ZC intersect YZ, XZ, and XY, respectively. Since $C = G$ is the centroid of $\triangle XYZ$, P, Q, and R are the midpoints of the sides on which they lie. Since $G = C$ is the circumcentre and P is the midpoint of YZ, it follows that CP, and hence XP, are perpendicular to YZ; similarly, $YQ \perp XZ$ and $ZR \perp XY$.



Since $|XP| = |XP|$, $\angle XPY = \angle XPZ$, and $|PY| = |PZ|$, we have $\triangle XPY \cong \triangle XPZ$, and so $|XY| = |XZ|$. Similarly, $|YX| = |YZ|$ and $|ZX| = |ZY|$. Thus $|XY| = |YZ| = |ZX|$, and so $\triangle XYZ$ is equilateral. ■

Quiz #11. *Monday, 7 December, 2015. [10 minutes]*

1. Suppose the Euler line of $\triangle XYZ$ is also the triangle's altitude from X. Show that $\angle XYZ = \angle ZXY$. *[5]*

SOLUTION. Suppose P is the foot of the altitude from X, *i.e.* the point where this altitude meets the (extension of the) side YZ. Since XP is an altitude, it is perpendicular to YZ, so $\angle XPY = \angle XPZ$. Since XP is also the Euler line, it passes through the centroid G of $\triangle XYZ$, so XP must also be a median. Thus P must be the midpoint of YZ, so $|PY| = |PZ|$. Since we obviously have $|XP| = |XP|$, it follows that $\triangle XPY \cong \triangle XPZ$ by the Side-Angle-Side congruence criterion. Thus $\angle XYZ = \angle XYP = \angle XZP = \angle ZXY$. ■