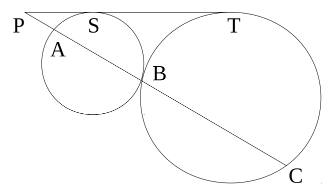
Mathematics 2260H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2015

Take-Home Final Examination
Due on Friday, 18 December, 2015.

Instructions: Do both of parts $\angle$ and $\odot$, and, if you wish, part Δ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part $\angle$. Do *all four* (4) of problems **1** – **4**. *[40 = 4 × 10 each]*

1. Suppose two circles are tangent to each other at B and also have a line tangent to both of them at points S and T respectively. Moreover, suppose P is a point on ST on the other side of S from T, and A and C are the other points where PB intersects each circle, as in the diagram below. Show that $|PT|^2 - |PS|^2 = |PB| \cdot |AC|$.



2. Determine, with proof, the maximum number of points which are all the same distance from each other that can be found in the Euclidean plane. How does the answer change if the points are in $\mathbb{R}^3$ instead?

3. A polygon is said to be *convex* if the line segments joining two vertices of the polygon never pass outside the polygon.

a. Show that the sum of the interior angles of a convex n-sided polygon is $(n-2)$ straight angles. *[5]*

b. Show that the sum of the interior angles of an arbitrary n-sided polygon is $(n-2)$ straight angles. *[5]*

4. A geometry $\mathcal{G}$ is defined as follows:

- The points of $\mathcal{G}$ are the points of the Cartesian plane inside the unit circle centred at the origin, *i.e.* $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 < 1$.
- The lines of $\mathcal{G}$ are the chords of the unit circle centred at the origin. (Not including the endpoints because points that are actually on the unit circle are not in $\mathcal{G}$.)
- Intersections, angles, distances, *etc.*, work as is usual in the Cartesian plane.

Determine, with proof, which of Euclid's five Postulates are true in $\mathcal{G}$.

[Parts $\odot$ and Δ are on page 2.]

Part $\odot$. Do any *four* (4) of problems **5** – **11**. *[40 = 4 × 10 each]*
Please draw the relevant diagram(s) in each case!

5. Suppose O is the circumcentre of $\triangle ABC$ and points X, Y, and Z are chosen so that BC is the perpendicular bisector of OX, AC is the perpendicular bisector of OY, and AB is the perpendicular bisector of OZ. Show that $\triangle XYZ \cong \triangle ABC$.

6. Suppose ℓ is a line and A and B are distinct points on one side of the line. Show that there is a circle passing through both A and B which is tangent to ℓ. Must such a circle be unique?

7. Suppose the incircle of $\triangle ABC$ touches AB at R, BC at P, and AC at Q. Show that AP, BQ, and CR are concurrent.

8. Suppose A, B, C, D, E, and F are points on a circle, arranged clockwise. Let U be the intersection of AE and BF, V be the intersection of AD and CF, and W be the intersection of BD and CE. Show that U, V, and W are collinear.

9. Join each vertex of $\triangle ABC$ to the points dividing the opposite side into equal thirds and let X, Y, and Z be the points of intersection of the pairs of these lines closest to BC, AC, and AB, respectively. Show that the sides of $\triangle XYZ$ are parallel to corresponding sides of $\triangle ABC$ and that $\triangle XYZ \sim \triangle ABC$.

10. Suppose $ABCD$ is a cyclic quadrilateral, *i.e.* A, B, C, and D are points on a circle, given in order going around the circle in some direction. Show that if we join each of A, B, C, and D to the orthocentre of the triangle formed by the other three, then the resulting line segments all intersect in a common midpoint M.

11. Suppose D, E, and F are points on the sides BC, AC, and AB of $\triangle ABC$, respectively, such that AD, BE, and CF all meet at a point P inside the triangle. Let Q be the point in which AD meets EF. Show that $|AQ| \cdot |PD| = |PQ| \cdot |AD|$.

[Total = 80]

Part $\triangle$. Bonus!

$\square$. Write an original poem about geometry. *[1]*

$\diamondsuit$. A chord PQ of a circle is tangent to a smaller circle with the same centre. Assuming that $|PQ| = 8$ *cm*, find the area of the region between the two circles. *[1]*

I HOPE THAT YOU ENJOYED THIS COURSE.
HAVE A GOOD BREAK!