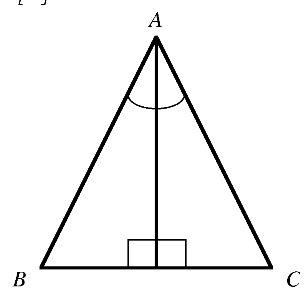
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2006

Quiz Solutions

Quiz #1. 22 September, 2006 *[5 minutes]*

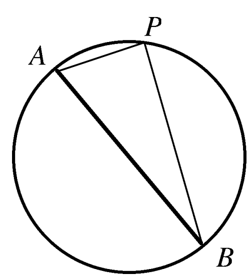
1. Suppose that the angle bisector of $\angle A$ in $\triangle ABC$ is also the altitude from vertex A. Show that $\triangle ABC$ is isosceles. *[5]*



Solution. Let Z be the point at which the bisector of $\angle A$ intersects BC. Since AZ bisects $\angle A$, it follows by definition that $\angle ZAB = \angle ZAC$. Observe that because the bisector of $\angle A$ is also the altitude from A, $\angle AZB = \angle AZC = 90°$, and note that it is trivial that $|AZ| = |AZ|$. Hence $\triangle AZB \cong \triangle AZC$ because the angle-side-angle criterion is satisfied. It follows that $|AB| = |AC|$, and so $\triangle ABC$ is isosceles by definition. ■

Quiz #2. 29 September, 2006 *[5 minutes]*

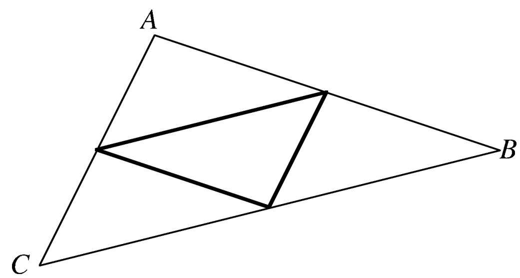
1. Suppose AB is a diameter of a circle and P is any other point on the circle. Show that $\angle APB = 90°$. *[5]*



Solution. $\angle APB$ is inscribed in a circle, and so must be equal to half of the arc that it subtends. Since AB is a diameter of the circle, the arc subtended by $\angle APB$ is half of the circle, *i.e.* 180°. Hence $\angle APB = 90°$. ■

Quiz #3. 6 October, 2006 *[5 minutes]*

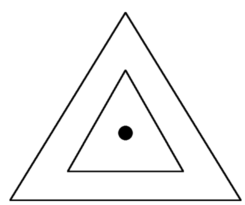
1. Show that the triangle whose vertices are the midpoints of the sides of $\triangle ABC$ (*i.e.* the *medial* triangle) is similar to $\triangle ABC$. *[5]*



Solution. Let X, Y, and Z be the midpoints of AB, BC, and AC respectively. It follows that $|XY| = \frac{1}{2}|AC|$, $|YZ| = \frac{1}{2}|AB|$, and $|XZ| = \frac{1}{2}|BC|$ (by Corollary 1.31). Hence the sides of $\triangle YZX$ are proportional to the sides of $\triangle ABC$, so $\triangle YZX \sim \triangle ABC$ (by Theorem 1.32). ■

Quiz #4. 13 October, 2006 *[5 minutes]*

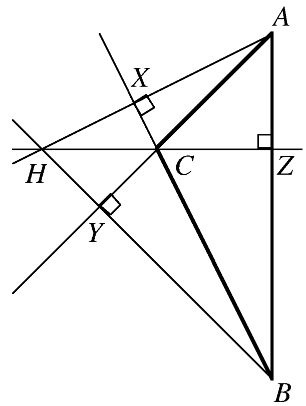
1. Give an example of triangles $\triangle ABC$ and $\triangle PQR$ which have the same circumcentre O and the same centroid G, but are *not* congruent.



Solution. Recall that the centroid and circumcentre coincide for any equilateral triangle. It follows that any two equilateral triangles of different sizes (hence not congruent) will serve as an example if positioned so that their centroids are at the same place. ■

Quiz #5. 20 October, 2006 *[5 minutes]*

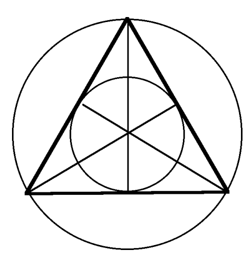
1. Suppose $\triangle ABC$ is *not* a right triangle and H is its orthocentre. Verify that C is the orthocentre of $\triangle ABH$. *[5]*



Solution. Let X, Y, and Z be the feet of the altitudes of $\triangle ABC$ from A, B, and C, respectively, so that H is the common point of intersection of the lines AX, BY, and CZ. Then, by the definition of altitude, it follows that $AX \perp BC$, $BY \perp AC$, and $CZ \perp AB$. Since the lines AH, BH, and CH are the same lines as AX, BY, and CZ, respectively, it now follows that $AH \perp BC$ (so $BC \perp AH$), $BH \perp AC$ (so $AC \perp BH$), and $CH \perp AB$ (so $HC \perp AB$). Hence the lines AC, BC, and HC each pass through one of the vertices of and are perpendicular to the opposite side of $\triangle ABH$. Thus these lines are the altitudes of $\triangle ABH$, and since they obviously all pass through C, it must be the orthocentre of $\triangle ABH$. ■

Quiz #6. 3 November, 2006 *[5 minutes]*

1. The centroid of an equilateral triangle is also its circumcentre and its incentre. What is the ratio of the circumradius of the triangle to the inradius? *[5]*

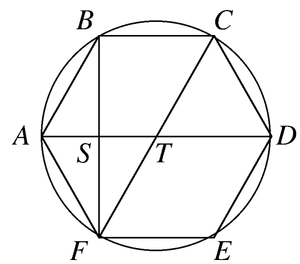


Solution. Suppose $\triangle ABC$ is equilateral, G is its centroid (also incentre and circumcentre), and the median (also altitude and angle bisector) from A meets BC at X. Then the inradius of $\triangle ABC$ is $|GX|$ and the circumradius is $|AG|$. However, we know that the centroid G lies two thirds of the way from A to X along the median AX (Theorem 2.7). Hence, the ratio of the circumradius of the triangle to the inradius is

$$\frac{|AG|}{|GX|} = \frac{2}{1} = 2\,. \qquad \blacksquare$$

Quiz #7. 10 November, 2006 *[5 minutes]*

1. Suppose $ABCDEF$ is a regular hexagon inscribed in a circle, and S and T are the intersections of BF and CF, respectively, with AD. Compute $\mathbf{cr}(A,S,T,D)$. *[5]*



Hint: The following values of $\sin(\theta)$ may be of some use:

θ	$\sin(\theta)$
$0°$	0
$30°$	$\frac{1}{2}$
$60°$	$\frac{1}{2}\sqrt{3}$
$90°$	1

Solution 1. Note that A, B, C, and D are on the circle and in persepective with the collinear points A, S, T, and D, respectively, from the point F on the circle. It follows that

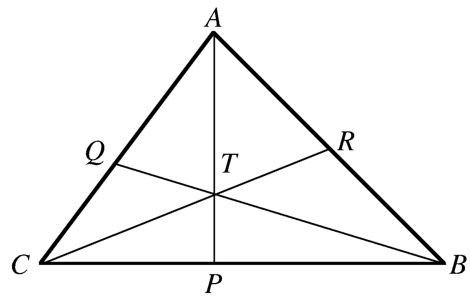
$$\begin{aligned}\mathbf{cr}(A,S,T,D) &= \mathbf{cr}(A,B,C,D)\\ &= \frac{\sin\left(\frac{1}{2}\mathrm{arc}(AC)\right)\sin\left(\frac{1}{2}\mathrm{arc}(BD)\right)}{\sin\left(\frac{1}{2}\mathrm{arc}(AD)\right)\sin\left(\frac{1}{2}\mathrm{arc}(BC)\right)}\\ &= \frac{\sin(60°)\sin(60°)}{\sin(90°)\sin(30°)}\\ &= \frac{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}}{1\cdot\frac{1}{2}}\\ &= \frac{3}{2}. \qquad \blacksquare\end{aligned}$$

Solution 2. Observe that AD and CF are diameters of the circle, and hence their intersection, T, is the centre of the circle. Since $ABCDEF$ is a regular hexagon each of its sides is equal to the radius of the circle (Why?) and it follows that BF bisects AT at S. This means that $|AS| = |ST|$, $|AT| = 2|ST|$, $|SD| = 3|ST|$, $|AD| = 4|ST|$. Hence

$$\mathbf{cr}(A,S,T,D) = \frac{|AT|\cdot|SD|}{|AD|\cdot|ST|} = \frac{2|ST|\cdot 3|ST|}{4|ST|\cdot|ST|} = \frac{3}{2}. \qquad \blacksquare$$

Quiz #8. 17 November, 2006 *[7 minutes]*

1. Suppose $\triangle ABC$ has $|AB| = 4\sqrt{2}$, $|AC| = 5$, and $|BC| = 7$. Assume that AP is the altitude from A and $|AP| = 4$, CR is the median from C, and Q is chosen on AC so that AP, BQ, and CR are concurrent. Determine $|QA|$. *[5]*



Solution. Since AP, BQ, and CR are concurrent, it follows by Ceva's Theorem that

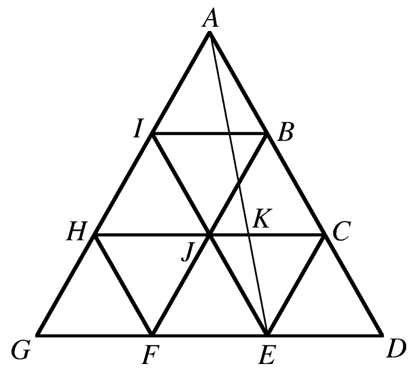
$$\frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = 1\,.$$

We have information about several parts of this product. First, since CR is the median from C, R is the midpoint of AB, *i.e.* $|AR| = |RB|$, so $\frac{|AR|}{|RB|} = 1$. Second, consider $\triangle APC$. We are given that $|AC| = 5$, $|AP| = 4$, and that $\angle APC = 90°$ (since AP is an altitude). By the Pythagorean Theorem, $25 = |AC|^2 = |AP|^2 + |PC|^2 = 16 + |PC|^2$, so $|PC| = \sqrt{25 - 16} = \sqrt{9} = 3$. It follows that $|BP| = |BC| - |PC| = 7 - 3 = 4$, and so $\frac{|BP|}{|PC|} = \frac{4}{3}$.

Plugging this information into the above product gives $1 \cdot \frac{4}{3} \cdot \frac{|CQ|}{|QA|} = 1$, *i.e.* $\frac{|CQ|}{|QA|} = \frac{3}{4}$, and hence $|CQ| = \frac{3}{4}|QA|$. On the other hand, $5 = |AC| = |CQ| + |QA| = \frac{3}{4}|QA| + |QA| = \frac{7}{4}|QA|$. Thus $|QA| = \frac{4}{7} \cdot 5 = \frac{20}{7}$. ■

Quiz #9. 24 November, 2006 *[7 minutes]*

1. Suppose A–J are the vertices of nine equilateral triangles arranged to form a large equilateral triangle as in the diagram, and suppose K is the point on JC such that $|JK| = \frac{1}{3}|JC|$. Show that A, K, and E are collinear. *[5]*



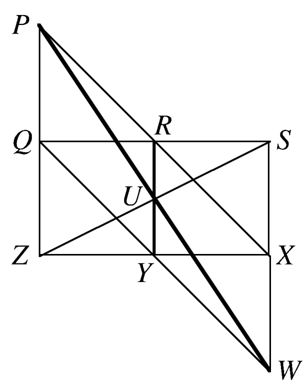
Solution. We will apply Menelaus' Theorem with respect to $\triangle HIJ$. Note that A is on the line HI and $\frac{|HA|}{|AI|} = \frac{2}{1}$, E is on the line IJ and $\frac{|IE|}{|EJ|} = \frac{2}{1}$, and K is on the line JH and $\frac{|JK|}{|KH|} = \frac{1/3}{4/3} = \frac{1}{4}$. Note also that all three of A, K, and E are not on $\triangle HIJ$. Thus

$$\frac{|HA|}{|AI|} \cdot \frac{|IE|}{|EJ|} \cdot \frac{|JK|}{|KH|} = \frac{2}{1} \cdot \frac{2}{1} \cdot \frac{1}{4} = \frac{4}{4} = 1\,,$$

and it follows by Menelaus' Theorem that A, K, and E are collinear, as desired. ■

Quiz #10. 1 December, 2006 *[7 minutes]*

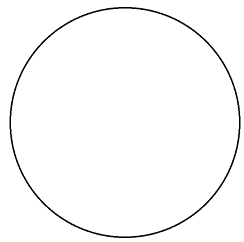
1. Suppose P–S and W–Z are the vertices of several adjacent congruent isosceles right-triangles as in the diagram, and suppose U is the point of intersection of SZ with RY. Show that P, U, and W are collinear. *[5]*



Solution. Note that the three points Q, R, and S are collinear, as are the three points X, Z, and Y. Observe that the lines QZ and XR intersect in P, the lines QY and XS intersect in W, and the lines RY and ZS intersect in U. It follows by Pappus' Theorem that the points P, U, and W are collinear. ■

Quiz #11. 7 December, 2006 *[5 minutes]*

1. Given a circle, find a ruler and compass construction which locates the centre of the circle. *[5]*



Solution. Here is a construction that does the job:

i. Pick any three distinct point A, B, and C on the given circle.

ii. Draw the line segments AB, AC, and BC connecting these three points to make $\triangle ABC$. Note that the given circle is then the circumcircle of $\triangle ABC$.

iii. Construct the perpendicular bisectors of each side of $\triangle ABC$ and extend them until they intersect at a common point. This common point is the centre of the given circle.

This construction works because the centre of the given circle is the circumcentre of $\triangle ABC$, which we know is the common point of intersection of the perpendicular bisectors of the sides of the triangle. ∎