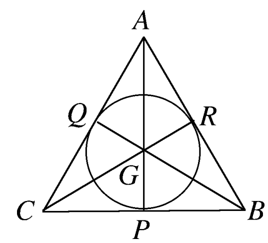
Mathematics 226H – Geometry I: Euclidean geometry

TRENT UNIVERSITY, Fall 2006

Solutions to Problem Set #7

1. (Exercise 2E.2) Suppose that the centroid and incentre of $\triangle ABC$ are the same point. Prove that the triangle is equilateral. *[5]*



Solution. Suppose AP, BQ, and CR are the medians of $\triangle ABC$ and G is the centroid. By Theorem 2.7, the centroid is two-thirds of the way along each median from the vertex to the opposite side, so each median is three times the length of the segment from the centroid to the opposite side. That is, $|AP| = 3|GP|$, $|BQ| = 3|GQ|$, and $|CR| = 3|GR|$. Since G is also the incentre, AP, BQ, and CR are also the angle bisectors of $\triangle ABC$. It follows that $|GP| = |GQ| = |GR|$, because these three line segments are all radii of the incircle, and hence the three medians are of equal length, *i.e.* $|AP| = |BQ| = |CR|$. Problem 2.8 now implies that $|AB| = |AC|$ because $|BQ| = |CR|$ and that $|AC| = |BC|$ because $|AP| = |BQ|$. Thus $|AB| = |AC| = |BC|$, so $\triangle ABC$ is equilateral. ■

2. (Exercise 2G.1) Join each vertex of $\triangle ABC$ to the points of trisection of the opposite side and let X, Y, and Z be the points of intersection of these side-trisectors, as shown in the figure below [or Figure 2.26 in the text]. Show that the sides $\triangle XYZ$ are parallel to corresponding sides of $\triangle ABC$ and that $\triangle XYZ \sim \triangle ABC$. *[5]*

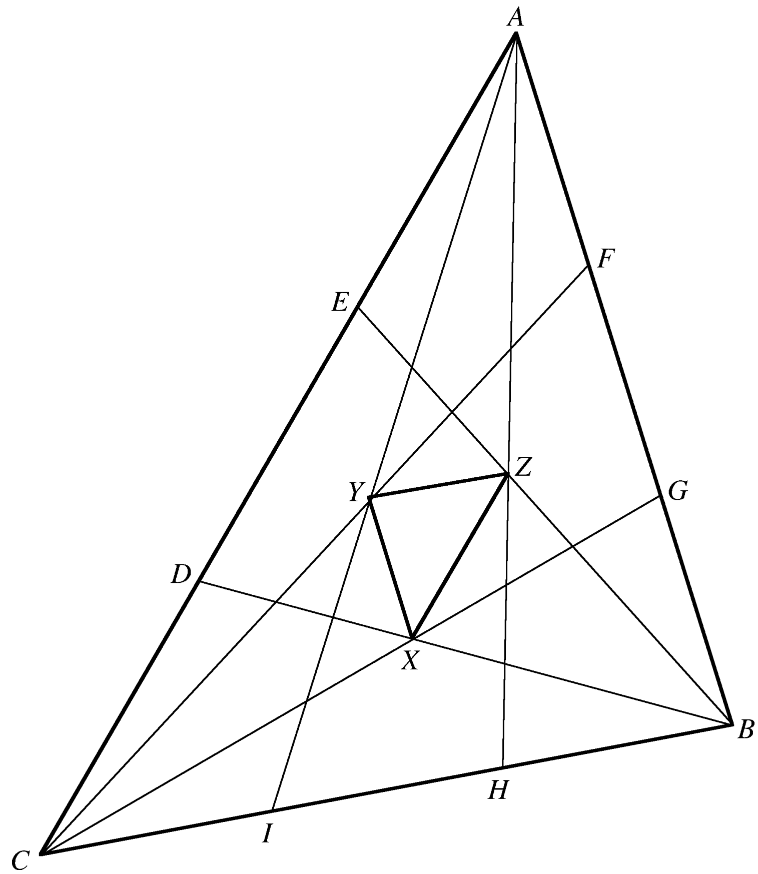
Solution. Let the points of trisection of the sides of $\triangle ABC$ be D, E, F, G, H, and I, respectively, if we move clockwise around the triangle from vertex C. (See the figure.) Then X, Y, and Z are the intersections of BD and CG, AI and CF, and AH and BE, respectively.

Note that because D and E trisect Ac and F and G trisect AB,

$$\frac{|AD|}{|AC|} = \frac{|AG|}{|AB|} = \frac{2}{3},$$

so $DG \parallel CB$ by Lemma 1.29. Since AC meets these parallel lines, it follows that $\angle ADG = \angle ACB$. As we also have that $\angle DAG = \angle CAB$ (because they are the same angle), it now follows from the angle-angle similarity criterion that $\triangle AGD \sim \triangle ABC$. This, by Theorem 1.28, implies that

$$\frac{|DG|}{|CB|} = \frac{|AD|}{|AC|} = \frac{|AG|}{|AB|} = \frac{2}{3}.$$



Since $DG \parallel CB$, the "Z-Theorem" implies that $\angle CBD = \angle BDG$ and $\angle BCG = \angle CGD$. The angle-angle similarity criterion then gives that $\triangle BCX \sim \triangle DGX$ and, since we also have $|DG|/|BC| = \frac{2}{3}$, it follows that

$$\frac{|DX|}{|XB|} = \frac{|GX|}{|XC|} = \frac{|DG|}{|BC|} = \frac{2}{3}\,.$$

Hence

$$|DB| = |DX| + |XB| = \frac{2}{3}|XB| + |XB| = \frac{5}{3}|XB| \qquad \text{and}$$
$$|GC| = |GX| + |XC| = \frac{2}{3}|XC| + |XC| = \frac{5}{3}|XC|\,,$$

, so

$$|XB| = \frac{3}{5}|DB| \qquad \text{and} \qquad |XC| = \frac{3}{5}|GC|\,.$$

Similar arguments allow us to conclude that

$$|YC| = \frac{3}{5}|FC|\,, \quad |YA| = \frac{3}{5}|IA|\,, \quad |ZA| = \frac{3}{5}|HA|\,, \quad \text{and} \quad |ZB| = \frac{3}{5}|EB|\,.$$

Since

$$\frac{|AY|}{|AI|} = \frac{|AZ|}{|AH|} = \frac{3}{5}\,,$$

it now follows from Lemma 1.29 that $YZ \parallel IH$, and hence that $YZ \parallel BC$ (since IH is a segment of BC). Also, since $\angle A$ is common to both triangles, Theorem 1.33 implies that $\triangle AIH \sim \triangle AYZ$, and it follows that

$$|YZ| = \frac{3}{5}|IH| = \frac{3}{5} \cdot \frac{1}{3}|BC| = \frac{1}{5}|BC|\,.$$

Similarly, $XY \parallel AB$, with $|XY| = \frac{1}{5}|AB|$, and $XZ \parallel AC$, with $|XZ| = \frac{1}{5}|AC|$. Since the sides of $\triangle XYZ$ are proportional to the corresponding sides of $\triangle ABC$, Theorem 1.32 implies that $\triangle XYZ \sim \triangle ABC$, as required. ■