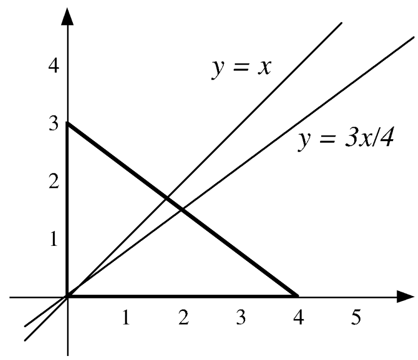
Mathematics 226H – Geometry I: Euclidean geometry
Trent University, Fall 2006

Solution to Problem Set #6

1. Suppose $\triangle ABC$ is not equilateral. Must the incentre of this triangle be on its Euler line? Prove it or give a counterexample. *[10]*



Solution. The incentre of a non-equilateral triangle need not be on its Euler line, as the following counterexample demonstrates.

Consider the 3-4-5 triangle in the Cartesian plane with vertices $A = (0,0)$, $B = (0,3)$, and $C = (4,0)$. Note that $\triangle ABC$ is a right triangle with the right angle at $A = (0,0)$. It is easy to see that sides AB and AC of the triangle are altitudes, and hence that the orthocentre of $\triangle ABC$ is the vertex $A = (0,0)$. It is also easy to check that the midpoint of BC, $\left(2, \frac{3}{2}\right)$, is equidistant from A, B, and C, and hence is the circumcentre of $\triangle ABC$. The Euler line is the line joining the orthocentre and circumcentre, which in this case has equation $y = \frac{3}{4}x$.

The incentre of $\triangle ABC$ is the intersection of the three angle bisectors of the triangle, so it must be on each of them. Observe that the bisector of the right angle at $A = (0,0)$ is obviously the line $y = x$. The only point at which the Euler line and the angle bisector at $A = (0,0)$ meet is $A = (0,0)$, but this cannot be the incentre as it is a vertex of the triangle. (The incenre has to be *inside* the triangle ...) Hence the incentre of $\triangle ABC$ is not on the Euler line. ■

Note: The argument above shows that the incentre of $\triangle ABC$ is not on the Euler line without ever explicitly locating the incentre.

Note the second: In an isosceles triangle the incentre does have to be on the Euler line. (Why?)