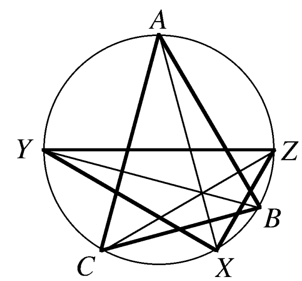
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2006

Problem Set #5

1. (Exercise 2A.3) Given acute angled $\triangle ABC$, extend the altitudes from A, B, and C to meet the circumcircle at points X, Y, and Z, respectively. Show that lines AX, BY, and CZ bisect the three angles of $\triangle XYZ$. *[5]*



Solution. Let H denote the orthocentre of $\triangle ABC$, the point at which the three altitudes from A, B, and C intersect. (Note that since $\triangle ABC$ is acute angled, H is inside the triangle.) Also, let S be the point at which AX intersects BC, T be the point at which BY intersects AC, and U be the point at which CZ intersects AB. We will show that AX bisects $\angle YXZ$.

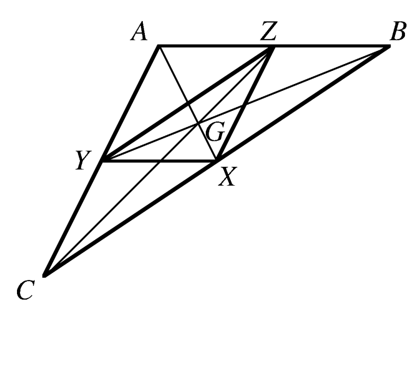
Observe that $\angle THC = \angle UHB$ (opposite angles) and $\angle CTH = 90° = \angle BUH$ (since BT and CU are altitudes of $\triangle ABC$). By the angle-angle similarity criterion, we have $\triangle CHT \sim \triangle BHU$, and hence $\angle TCH = \angle UBH$. It now follows, using Theorem 1.16, that

$$\angle AXZ = \frac{1}{2}\text{arc}(AZ) = \angle ACZ = \angle TCH = \angle UBH = \angle ABY = \frac{1}{2}\text{arc}(AY) = \angle AXY\,.$$

Thus AX does bisect $\angle YXZ$, the angle at X of $\triangle XYZ$.

Similar arguments show that BY and CZ bisect the angles at Y and Z, respectively, of $\triangle XYZ$. ■

2. (Exercise 2B.1) Show that the centroid of the medial triangle of $\triangle ABC$ is the centroid of $\triangle ABC$. *[5]*



Solution. Let X, Y, and Z be the midpoints of the sides BC, AC, and AB, respectively, of $\triangle ABC$. Now let G be the centroid of $\triangle ABC$; by definition, this means that G is the common point of intersection of the medians AX, BY, and CZ. To show that G is also the centroid of $\triangle XYZ$ it is sufficient to show that that the medians of $\triangle ABC$ are also medians of $\triangle XYZ$. That is, we need to verify that AX bisects YZ, BY bisects XZ, and CZ bisects XY.

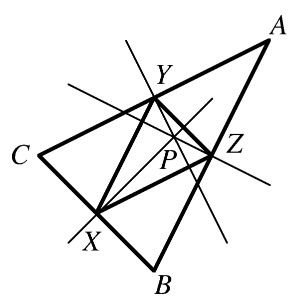
Note that by Corollary 1.31, $XY \parallel AB$, $YZ \parallel BC$, and $XZ \parallel AC$, and also $|XY| = \frac{1}{2}|AB|$, $|YZ| = \frac{1}{2}|BC|$, and $|XZ| = \frac{1}{2}|AC|$. (By the side-side-side criterion for similarity, it follows from the latter facts that $\triangle XYZ \sim \triangle ABC$.) Let P be the point of intersection of AX with YZ. We will show that $\triangle AXC \sim \triangle XPZ$.

Since YZ is a transversal between the parallel lines AC and XZ and $\angle AYZ = \angle XZY$. As AC is a transversal between the parallel lines YZ and BC, it now follows $\angle ACX = \angle AYZ = \angle ZXY$. Since AX is another transversal between the parallel lines AC and XZ, we also have that $\angle XAP = \angle PXZ$. Hence, by the angle-angle criterion for similarity of triangles, $\triangle AXC \sim \triangle XPZ$.

Because these triangles are similar, it follows that $\frac{|PZ|}{|XC|} = \frac{|XZ|}{|AC|} = \frac{1}{2}$, *i.e.* $|PZ| = \frac{1}{2}|XC| = \frac{1}{2}\left(\frac{1}{2}|BC|\right) = \frac{1}{2}|YZ|$. Thus AX bisects YZ at P, *i.e.* AX is also a median of $\triangle XYZ$.

Similar arguments can be used to show that the other medians of $\triangle ABC$ are also medians of $\triangle XYZ$, from which it follows that the centroid of $\triangle ABC$ is also the centroid of its medial triangle, $\triangle XYZ$. ■

3. (Exercise 2C.2) Show that the circumcentre of $\triangle ABC$ is the orthocentre of the medial triangle. *[5]*



Solution. Let X, Y, and Z be the midpoints of the sides BC, AC, and AB, respectively, of $\triangle ABC$. The circumcentre P of $\triangle ABC$ is the common intersection of the perpendicular bisectors of the sides BC, AC, and AB. Note that by Corollary 1.31, $XY \parallel AB$, $YZ \parallel BC$, and $XZ \parallel AC$. It follows that $XP \perp YZ$, $YP \perp XZ$, and $ZP \perp XY$, and hence the lines XP, YP, and ZP are the altitudes of $\triangle XYZ$ (from X, Y, and Z, respectively). Since these altitudess all intersect at P, this point is the orthocentre of $\triangle XYZ$, the medial triangle of $\triangle ABC$. ■