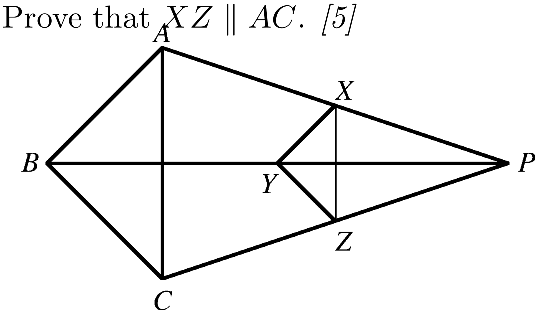
**Mathematics 226H – Geometry I: Euclidean geometry**
Trent University, Fall 2006

**Solutions to Problem Set #4**

**1.** (Exercise 1H.6) In the figure below [Figure 1.45 in the text], line segments $PA$, $PB$, and $PC$ join point $P$ to the three vertices of $\triangle ABC$. We have chosen point $Y$ on $PB$ and drawn $YX$ and $YZ$ parallel to $BA$ and $BC$, respectively, where $X$ lies on $PA$ and $Z$ lies on $PC$. Prove that $XZ \parallel AC$. *[5]*



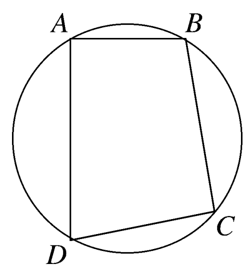

**Solution.** $\triangle ABP$ and $\triangle XYP$ have a common angle at $P$, so $\angle APB = \angle XPY$. Since $AB \parallel XY$ and $AP$ crosses $XY$ at $X$, and corresponding angles of a transversal to two parallel lines are equal, we also have that $\angle PAB = \angle PXY$. By the angle-angle similarity criterion it follows that $\triangle ABP \sim \triangle XYP$. A similar argument to the one above shows that $\triangle CBP \sim \triangle ZYP$.

Since $BP$ is a common side of $\triangle ABP$ and $\triangle CBP$ and $YP$ is a common side of $\triangle XYP$ and $\triangle ZYP$, the scaling factors in the above pairs of similar triangles are the same. In particular, $\frac{|PX|}{|PA|} = \frac{|PY|}{|PB|} = \frac{|PZ|}{|PC|}$. Applying this fact to $\triangle ACP$, we get that $XZ \parallel AC$ by Lemma 1.29. ■

**2.** (Exercise 2A.1) Show that quadrilateral $ABCD$ can be inscribed in a circle if and only if $\angle B$ and $\angle D$ are supplementary. *[5]*

*Hint:* To prove if, show that $D$ lies on the unique circle through $A$, $B$, and $C$.

*Note:* A quadrilateral inscribed in a circle is said to be *cyclic*.




**Solution.** We will prove the two directions of the "if and only if" separately.

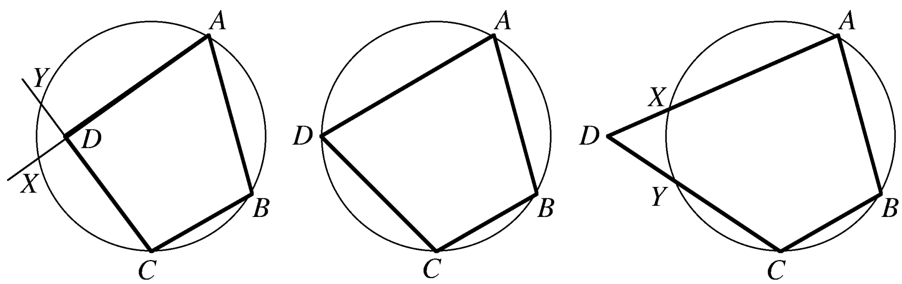
($\Longrightarrow$ [*i.e.* "only if"]) Suppose that quadrilateral $ABCD$ can be inscribed in a circle. Then $\angle B = \frac{1}{2}\text{arc}(ADC)$ and $\angle D = \frac{1}{2}\text{arc}(ABC)$ by Theorem 1.16. However, the two arcs make up the whole circle between them, so:

$$\angle B + \angle D = \frac{1}{2}\text{arc}(ADC) + \frac{1}{2}\text{arc}(ABC) = \frac{1}{2}\left(\text{arc}(ADC) + \text{arc}(ABC)\right) = \frac{1}{2} \cdot 360^\circ = 180^\circ$$

Hence $\angle B$ and $\angle D$ are supplementary.

($\Longleftarrow$ [*i.e.* "if"]) Suppose $ABCD$ is a quadrilateral such that $\angle B$ and $\angle D$ are supplementary. Consider the possible locations of $D$ with respect to the unique circle passing through $A$, $B$, and $C$. There are three cases:

*i.* $D$ is inside the circle.
*ii.* $D$ is on the circle.
*iii.* $D$ is outside the circle.




*i.* "In" *ii.* "On" *iii.* "Out"

In case *ii*, of course, we're done. We will show that cases *i* and *iii* cannot occur. In each of these two cases, let $X$ and $Y$ be the intersections of $AD$ and $CD$, respectively, with the circle passing through $A$, $B$, and $C$. In case *i*, of course, we have to extend $AD$ and $CD$ to meet the circle in order to locate $X$ and $Y$. Note that in both cases $\angle B = \angle ABC = \frac{1}{2}\text{arc}(AXC)$ by Theorem 1.16.

In case *i*, it follows from Corollary 1.19 that $\angle D = \frac{1}{2}\left(\text{arc}(ABC) + \text{arc}(XY)\right)$, so

$$\begin{aligned}\angle B + \angle D &= \frac{1}{2}\text{arc}(AXC) + \frac{1}{2}\left(\text{arc}(ABC) + \text{arc}(XY)\right) \\ &= \frac{1}{2}\left(\text{arc}(ABC) + \text{arc}(AXC)\right) + \frac{1}{2}\text{arc}(XY) \\ &= \frac{1}{2}360^\circ + \frac{1}{2}\text{arc}(XY) > 180^\circ .\end{aligned}$$

This contradicts the fact that $\angle B$ and $\angle D$ are supplementary, so case *i* cannot occur.

In case *iii*, it follows from Corollary 1.18 that $\angle D = \frac{1}{2}\left(\text{arc}(ABC) - \text{arc}(XY)\right)$, so

$$\begin{aligned}\angle B + \angle D &= \frac{1}{2}\text{arc}(AXC) - \frac{1}{2}\left(\text{arc}(ABC) + \text{arc}(XY)\right) \\ &= \frac{1}{2}\left(\text{arc}(ABC) + \text{arc}(AXC)\right) - \frac{1}{2}\text{arc}(XY) \\ &= \frac{1}{2}360^\circ - \frac{1}{2}\text{arc}(XY) < 180^\circ .\end{aligned}$$

This contradicts the fact that $\angle B$ and $\angle D$ are supplementary, so case *iii* also cannot occur.

Since cases *i* and *iii* contradict the premiss that $\angle B$ and $\angle D$ are supplementary, it follows that case *ii* must hold, *i.e.* that $D$ is on the circle passing through $A$, $B$, and $C$. Hence the quadrilateral $ABCD$ can be inscribed in a circle. ■