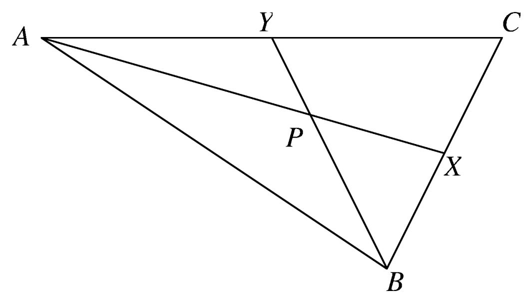
Mathematics 226H – Geometry I: Euclidean geometry
TRENT UNIVERSITY, Fall 2006

Solutions to Problem Set #2

1. (Exercise 1E.1) Draw two medians of a triangle. This subdivides the interior of the triangle into four pieces: three triangles and a quadrilateral. Show that two of the three small triangles have equal area and that the area of the third is equal to that of the quadrilateral. *[5]*

Solution. Let the vertices of our triangle be A, B, and C, and let AX and BY be the medians from A to BC and from B to AC respectively. Observation that this means that $|BX| = |XC| = \frac{1}{2}|BC|$ and $|AY| = |YC| = \frac{1}{2}|AC|$.



It follows from the observation above that $\triangle ABY$ has the same height and half the base of $\triangle ABC$, and so $K_{ABY} = \frac{1}{2}K_{ABC}$. Similarly, $\triangle BAX$ has the same height and half the base of $\triangle BAC = \triangle ABC$, and so $K_{BAX} = \frac{1}{2}K_{ABC}$. Since $\triangle ABY = \triangle APY + \triangle ABP$ and $\triangle BAX = \triangle BPX + \triangle ABP$, we get:

$$K_{APY} + K_{ABP} = K_{ABY} = \frac{1}{2}K_{ABC} = K_{BAX} = K_{BPX} + K_{ABP}$$

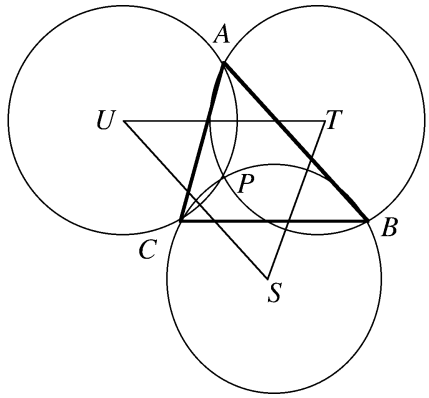
Subtracting K_{ABP} from both the left and right ends above gives us $K_{APY} = K_{BPX}$, *i.e.* the small triangles $\triangle APY$ and $\triangle BPX$ have equal area.

A very similar argument (details left to you!) shows that the remaining small triangle, $\triangle ABP$, has the same area as the quadrilateral $CXPY$. ■

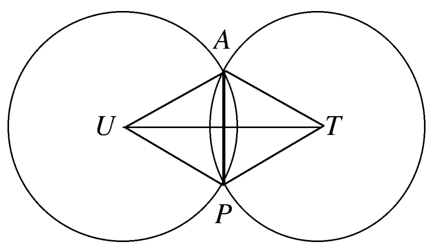
2. (Exercise 1F.11) Suppose three circles of equal radius go through a common point P, and denote by A, B, and C the three other points where [two] of these circles cross. Show that the unique circle through A, B, and C has the same radius as the original three circles. *[5]*

Hint: Use Exercise 1D.12 to show that $\triangle ABC$ is congruent to the triangle formed by the centers of the three given circles. Use the fact that circumcircles of congruent triangles have equal radii.

Solution. Let S, T, and U be the centres of the three circles, as in the diagram below. Following the hint, we will use Exercise 1D.12 to show that $\triangle ABC \cong \triangle SUT$. Note first that P is a point on each of the three original circles of equal radius, so it is a distance of one common radius from each of the centres of these circles, *i.e.* it is equidistant from the vertices of $\triangle SUT$.



We claim that A is the reflection of P in side UT of $\triangle SUT$, *i.e.* UT is a perpendicular bisector of AP. To show this, consider the following, which reproduces the relevant part of the preceding diagram with some additional lines.



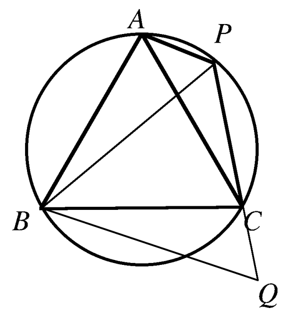
UA and UP are both radii of the circle centred on U, and TA and TP ae both radii of the circle centred on T. Since these circles have equal radii, we have that $|UA| = |UP| = |TA| = |TP|$, so $\triangle AUT$ and $\triangle PUT$ are both isosceles. Since these triangles share the common side UT, it follows, by the side-side-side criterion, that they are congruent. It follows that $\angle AUT = \angle PUT$, so UT is a bisector of $\angle AUP$. Since $|UA| = |UP|$, $\triangle AUP$ is isosceles, so the bisector of $\angle AUP$ is also a median and and an altitude, *i.e.* UT is a perpendicular bisector of the base AP of $\triangle AUP$, as desired.

Similar arguments can be used to show that B is the reflection of P in side TS of $\triangle SUT$ and C is the reflection of P in side SU of $\triangle SUT$. $\triangle ABC$ and $\triangle SUT$ thus satisfy the hypotheses of 1D.12, from which it follows that $\triangle ABC \cong \triangle SUT$. Since P is equidistant from S, U, and T, the circumcircle of $\triangle SUT$ must have centre P and radius $|SP| = |TP| = |UP|$, which is the common radius of the original three circles. As $\triangle ABC \cong \triangle SUT$, the circumcircle of $\triangle ABC$, which is the unique circle through A, B, and C, must have the same radius. ■

3. (Exercise 1G.3) An equilateral triangle ABC is inscribed in a circle and point P is chosen arbitrarily on the arc between A and C. Show that $|AP| + |PC| = |PB|$. *[5]*

Hint: Extend chord PC to point Q so that $|PQ| = |PB|$ and then draw BQ.

Solution. Following the hint, extend chord PC to point Q so that $|PQ| = |PB|$ and then draw BQ.



We will show that $|CQ| = |AP|$ by verifying that $\triangle PAB \cong \triangle QCB$. The first step will be to show that $\triangle BPQ$ is equilateral:

Since $\angle BAC$ and $\angle BPC$ both subtend the same arc of the circle, namely arc(BC), they must be equal, *i.e.* $\angle BPC = \angle BAC = 60°$. (For the last equality, recall that $\triangle ABC$ is equilateral.) It follows that $\triangle BPQ$ is equilateral: it satisfies the side-angle-side congruence criterion with any equilateral triangle with sides of length $|PQ| = |PB|$.

We will show that $\triangle PAB \cong \triangle QCB$ using the side-angle-side congruence criterion. Note that $|AB| = |CB|$ because $\triangle ABC$ is equilateral and $|BP| = |BQ|$ because $\triangle BPQ$ is equilateral. We still need to verify that $\angle ABP = \angle CBQ$. Observe that because $\triangle ABC$ and $\triangle PBQ$ are both equilateral, $\angle ABC = \angle PBQ = 60°$. It follows that:

$$\angle ABP + \angle PBC = \angle ABC = 60° = \angle PBQ = \angle PBC + \angle CBQ$$

Subtracting $\angle PBQ$ from both ends gives $\angle ABP = \angle CBQ$, as desired. Hence $\triangle PAB \cong \triangle QCB$ by the side-angle-side criterion.

Since $\triangle PAB \cong \triangle QCB$, $|CQ| = |AP|$, so $|AP| + |PC| = |CQ| + |PC| = |PB|$. ■

Bonus. (Exercise 1E.3) Since a triangle is determined by angle-side-angle, there should be a formula for K_{ABC} expressed in terms of a [the side opposite vertex A] and $\angle B$ and $\angle C$. Derive such a formula. *[2]*

Solution. I'm too lazy to do the bonus … ■