The Zermelo-Fraenkel Axioms of Set Theory

These axioms are the most common ones used for set theory these days. They are somewhat informally stated: a formal statement would require specifying a formal first-order language for set theory, defining various auxiliary ideas, such as subsets, and then developing their basic properties from some of the axioms before using them to state other axioms.

0. *Empty Set Axiom.* There is a set, usually denoted by $\emptyset$, which has no elements.

1. *Axiom of Extensionality.* If two sets x and y have exactly the same elements, *i.e.* for all a we have $a \in x$ if and only if $a \in y$, then $x = y$.

2. *Pair Set Axiom.* If x and y are sets, then $\{x, y\}$ is a set.

3. *Union Axiom.* If x is set of sets, then then the union of these sets, $\bigcup x = \bigcup_{a \in x} a = \{\, z \mid z \in a \text{ for some } a \in x \,\}$, is a set.

4. *Power Set Axiom.* If x is a set, then the collection of all subsets of x, $\mathcal{P}(x) = \{\, y \mid y \subseteq x \,\}$, is a set.

5. *Axiom of Infinity.* Let $S(x) = x \cup \{x\}$ be the *successor* operation on sets. Then the collection of all the successors of the empty set, $\omega = \mathbb{N} = \{\, \emptyset,\, S(\emptyset),\, S\left(S(\emptyset)\right),\, S\left(S\left(S(\emptyset)\right)\right), \dots \}$, is a set.

6. *Axiom of Foundation.* (Often called the *Axiom of Regularity.*) If x is any non-empty set, then there is an element $a \in x$ such that $a \cap x = \emptyset$.

7. *Axiom of Replacement.* Suppose f is a function (definable by some formula of the language of set theory) with domain x. Then the range of x, $\{\, f(a) \mid a \in x \,\}$, is a set.

8. *Axiom of Comprehension.* If x is a set, then every definable subset of x is a set.* That is, if $\varphi(y)$ is a formula of the language of set theory with parameter y, and A is a set, then $\{\, a \in A \mid \varphi(a) \text{ is true} \,\}$ is a set.

Set theory with this set of axioms is usually called Zermelo-Fraenkel set theory, or just ZF, after its creators. Note that this set of axioms is a compromise between minimality and convenience. For example, the Empty Set and Pair Set Axioms are deducible from the other axioms, but it's a modest pain to have to bother doing so.

One additional axiom is usually included:

9. *Axiom of Choice.* Suppose x is a set of non-empty sets. Then there is a function $f : x \to \bigcup x$ such that for all $a \in x$, $f(a) \in a$. (f is said to be a *choice function* for x.)

The Axiom of Choice, often referred to as just AC, is only necessary for dealing with infinite sets, as it is provable from the axioms of ZF for finite sets x, and its full power is rarely needed. However, we do not know, for example, how to define and develop calculus without using at least some of the power of this axiom. Set theory using the Zermelo-Fraenkel system of axioms together with the Axiom of Choice is often referred to as ZFC.

References

1. *The Joy of Sets: Fundamentals of Contemporary Set Theory* (2nd Edition), by Keith Devlin, 1992. Springer-Verlag, New York, ISBN 0-387-94094-4.

2. *An Outline of Set Theory*, by James M. Henle, 1986. Springer-Verlag, New York, ISBN 0-387-96368-5. (Reprinted since then by Dover Publications.)

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* Letting only definable collections that are subsets of existing sets be sets is how this system of axioms avoids Russell's Paradox. If, for example, the collection $R = \{\, x \mid x \text{ is a set and } x \notin x \,\}$ were allowed to be a set, then we would have $R \in R \iff R \notin R$, which is a contradiction.