Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Final Examination
Due on Friday, 12 December.
(Via Blackboard, on paper, or by email as a last resort.)

Instructions: Do both of parts $\in$ and $\subseteq$, and, if you wish, part $=$ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, and anything on or linked to from Blackboard, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, *you may not consult any other sources, nor consult or work with any other person on this exam.*

Part $\in$. Do any *four* (4) of problems **1** – **5**. *[40 = 4 × 10 each]*

1. Suppose $<_A$ is a (strict) linear order on a set A with the property that every non-empty subset $B \subseteq A$ has both a greatest element and a least element in B itself. Show that A must be a finite set. *[10]*

2. Suppose $n > 1$ is a natural number such that $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, and $r = 9 \cdot 2^{2n-1} - 1$ are all prime numbers. Show that $a = p \cdot q \cdot 2^n$ and $b = r \cdot 2^n$ are a pair of *amicable numbers*, that is, each is the sum of the other's divisors (other than the other itself). *[10]*

3. Suppose $P(x)$ is a one-place relation in a first-order language. Write a formula in the language that expresses the statement "There are exactly three possible values of x for which $P(x)$ is true." *[10]*

4. Suppose that for each $n \in \mathbb{N}$, A_n is an infinite and countable set, and that $A_i \cap A_j = \emptyset$ if $i \neq j$. Show that the union of all the A_n, $A = \bigcup_{n=0}^{\infty} A_n$, is also countable. *[10]*

5. Show that the right cancellation law for multiplication, $a \cdot_{\mathbb{N}} b = c \cdot_{\mathbb{N}} b$ and $b \neq 0_{\mathbb{N}}$ imply that $a = c$, is true in the natural numbers. *[10]*

Part $\subseteq$. Do any *four* (4) of problems **6** – **15**. *[40 = 4 × 10 each]*

6. The Island of Knights and Knaves has only those two kinds of inhabitants. Knights always tell the truth and knaves always lie. You meet nine inhabitants: Sue, Bob, Mel, Marge, Zoey, Homer, Betty, Carl and Sally. Sue says that Carl is a knave. Bob tells you that neither Marge nor Mel are knaves. Mel says, "I know that Marge is a knave and that Sally is a knight." Marge says that Homer is a knave. Zoey tells you that both Bob is a knight and Marge is a knave. Homer says that Carl is a knave or Sue is a knight. Betty tells you that Sue is a knave and Carl is a knight. Carl tells you, "Sally and Bob are knights." Sally tells you that Zoey could claim that Homer is a knave.

Determine which of the nine is a knight and which is a knave. *[10]*

7. *Pentominoes* are shapes obtained by gluing five 1×1 squares together full edge to full edge. Two pentominoes that can be made congruent via reflections (*i.e.* flips) or rotations are considered to be the same. Find all twelve pentominoes and an arrangement of all of them into a 5×12 rectangle. *[10]*

8. Show that every natural number n is equal to a sum of the form

$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1! + a_0 \cdot 0!$$

for some $k \geq 0$ and such that each a_i is a natural number with $0 \leq a_i \leq i$. *[10]*

9. Define the logical connective $\uparrow$ via the following truth table:

A	B	$A \uparrow B$
T	T	F
T	F	T
F	T	T
F	F	T

a. Write a formula truth-table equivalent to $A \uparrow B$ using the logical connectives $\neg$ and $\rightarrow$. *[2]*

b. Write formulas using just the connective $\uparrow$ that are truth-table equivalent to $\neg A$, $A \vee B$, $A \wedge B$, and $A \rightarrow B$. *[8]*

NOTE: In both parts **a** and **b** you may use the connective(s) you are supposed to use as many times as you like in the desired formula.

10. Show that the set ${}^{\mathbb{N}}2 = \{\, f \mid f : \mathbb{N} \rightarrow \{0,1\} \text{ is a function} \,\}$ is uncountable. *[10]*

11. Suppose r is positive real number. Officially, r is a schnitt. Use it to define the schnitt that is the real number $\frac{1}{r}$. *[10]*

12. In a certain mathematics class Professor B, who always tells the truth and is never mistaken†, explains the marking scheme for the course to the students at the start of the term.

> "This course meets once each week. There will be only one test, which will be written in class in one of the next twelve weeks. However, you will not know which week it is until the class in which the test is given."

Is there any way to determine in which week the test is given? Explain why or why not. If there is, in which week will the test be written? *[10]*

13. Suppose that x is a finite set that is well-ordered by $\in$ and is downward closed, *i.e.* if $a \in x$ and $b \in a$, then $b \in x$. Show that $x \in \mathbb{N}$. *[10]*

14. Suppose that $k = a^2 + b^2$ and $n = c^2 + d^2$ for some a, b, c, $d \in \mathbb{N}$. Show that $kn = e^2 + f^2$ for some e, $f \in \mathbb{N}$. *[10]*

15. Suppose p is a prime number. Show that $\sqrt{p}$ is irrational. *[10]*

[Total = 80]

Part =. Bonus questions!

$\mathbf{4^2 + 0}$. Write an original poem about logic or mathematics. *[0.5]*

$\mathbf{2^4 + 1}$. Is $p(n) = n^2 - n + 41$ a prime number for every $n \in \mathbb{N}$? Either way, prove it. *[0.5]*

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY THE BREAK!

† Yes, please *do* suspend your disbelief! :-)