Mathematics 2200H – Mathematical Reasoning
TRENT UNIVERSITY, Fall 2025

Assignment #7
Well-Orders

Due on Friday, 7 November.[*]

Recall that a (strict) linear order on a set A, let's denote it by $\triangleleft$, is a binary relation satisfying the following conditions:

1. *Irreflexivity:* For all $a \in A$, it is not the case that $a \triangleleft a$.
2. *Transitivity:* For all a, b, $c \in A$, if $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.
3. *Trichotomy:* For all a, $b \in A$, exactly one of $a \triangleleft b$, $a = b$, or $b \triangleleft a$, is true.

It is a *well-order* if it also satisfies the following condition:

4. *Least element principle:* If X is a non-empty subset of A, then X has a least element, *i.e.* there is a $y \in X$ such that for all $x \in X$, $y = x$ or $y \triangleleft x$.

1. Show that $<_{\mathbb{Z}}$ is not a well-order on the integers. *[1]*

2. Show that $<_{\mathbb{N}}$ is indeed a well-order on the natural numbers. (Note that we already know it is a linear order.) *[5]*

Another possibility for a linear order $\triangleleft$ on a set A to satisfy is the *Descending chain condition*:

> Every strictly decreasing sequence of elements of A, say a_0, a_1, a_2, ... *i.e.* $a_{k+1} \triangleleft a_k$ for each k, is finite, *i.e.* it ends at a_n for some $n \in \mathbb{N}$.

3. Show that a linear order satisfies the least element principle if and only if it satisfies the descending chain condition. *[4]*

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[*] Please submit your solutions, preferably as a single pdf, via Blackboard's Assignments module. If that fails, please submit them to the instructor on paper or via email to `sbilaniuk@trentu.ca` as soon as you can,