Mathematics 2200H – Mathematical Reasoning
Trent University, Fall 2025

Solutions to Assignment ~~#7~~ #8
Well-Orders
Due on Friday, 7 November.

Recall that a (strict) linear order on a set A, let's denote it by $\triangleleft$, is a binary relation satisfying the following conditions:

1. *Irreflexivity:* For all $a \in A$, it is not the case that $a \triangleleft a$.
2. *Transitivity:* For all a, b, $c \in A$, if $a \triangleleft b$ and $b \triangleleft c$, then $a \triangleleft c$.
3. *Trichotomy:* For all a, $b \in A$, exactly one of $a \triangleleft b$, $a = b$, or $b \triangleleft a$, is true.

It is a *well-order* if it also satisfies the following condition:

4. *Least element principle:* If X is a non-empty subset of A, then X has a least element, *i.e.* there is a $y \in X$ such that for all $x \in X$, $y = x$ or $y \triangleleft x$.

1. Show that $<_{\mathbb{Z}}$ is not a well-order on the integers. *[1]*

Solution. $\mathbb{Z}^- = \{ -1, -2, -3, \dots \}$ is a non-empty subset of $\mathbb{Z}$ which has no least element. Thus $<_{\mathbb{Z}}$ does not satisfy the least element principle, and so is not a well-order. ■

2. Show that $<_{\mathbb{N}}$ is indeed a well-order on the natural numbers. (Note that we already know it is a linear order.) *[5]*

Solution. Since we already know that $<_{\mathbb{N}}$ is indeed a linear-order on on $\mathbb{N}$, we only have to check that it satisfies the least element principle. Recall that $a <_{\mathbb{N}} b \iff a \in b$ by question **2** of Assignment #6.

Suppose $X \subseteq \mathbb{N}$ and $X \neq \emptyset$. By the Axiom of Foundation, there is a $k \in X$ such that $k \cap X = \emptyset$. We claim that k is the least element of X.

If $k = 0$, it is the least element of X because 0 is the least element in all of $\mathbb{N}$.

If $k \neq 0$, then $k = S(m) = \{ 0, 1, \dots, m \}$ for some $m \in \mathbb{N}$ and $k \cap X = \{ 0, 1, \dots, m \} \cap X = \emptyset$. It follows that no $n <_{\mathbb{N}} k \iff n \in k$ is in X. By trichotomy, it follows in turn that for every $x \in X$, either $x = k$ or $k <_{\mathbb{N}} x$. Thus k is the least element of X.

Either way, every non-empty $X \subset \mathbb{N}$ has a least element, so $<_{\mathbb{N}}$ satisfies the least element principle and thus is a well-order. ■

Another possibility for a linear order $\triangleleft$ on a set A to satisfy is the *Descending chain condition*:

> Every strictly decreasing sequence of elements of A, say a_0, a_1, a_2, … *i.e.* $a_{k+1} \triangleleft a_k$ for each k, is finite, *i.e.* it ends at a_n for some $n \in \mathbb{N}$.

3. Show that a linear order satisfies the least element principle if and only if it satisfies the descending chain condition. *[4]*

Solution. ($\Longrightarrow$) Assume that the linear order $\triangleleft$ on a set A satisfies the least element principle and suppose a_0, a_1, a_2, … is a strictly decreasing sequence of elements of A. By the least element principle, $S = \{ a_0, a_1, a_2, \dots \}$ must have a least element, say a_n. Since the sequence is strictly descending, it can have no further elements, so it is finite.

Thus if the linear order $\triangleleft$ on a set A satisfies the least element principle, it must also satisfy the descending chain condition.

($\Longleftarrow$) Assume that the linear order $\triangleleft$ on a set A satisfies the descending chain condition, but, by way of contradiction, there is some X with $\emptyset \neq X \subseteq A$ which does not have a least element. Construct a strictly descending seqence of elements of X as follows:

- Let a_0 be any element of X.
- Given that $a_k \in X$ has been chosen and X has no least element, we can choose an $a_{k+1} \in X$ such that $a_{k+1} \triangleleft a_k$.

Since X has no least element, this process never terminates. This means that a_0, a_1, a_2, ... is an infinite strictly decreasing sequence of elements of A, contradicting the descending chain condition.

Thus if the linear order $\triangleleft$ on a set A satisfies the descending chain condition, it must also satisfy the least element principle. ■